

# KEROV'S CENTRAL LIMIT THEOREM FOR THE PLANCHEREL MEASURE ON YOUNG DIAGRAMS

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*In memory of Sergei Kerov (1946–2000)*

**Abstract.** Consider random Young diagrams with fixed number  $n$  of boxes, distributed according to the Plancherel measure  $M_n$ . That is, the weight  $M_n(\lambda)$  of a diagram  $\lambda$  equals  $\dim^2 \lambda/n!$ , where  $\dim \lambda$  denotes the dimension of the irreducible representation of the symmetric group  $\mathfrak{S}_n$  indexed by  $\lambda$ . As  $n \rightarrow \infty$ , the boundary of the (appropriately rescaled) random shape  $\lambda$  concentrates near a curve  $\Omega$  (Logan–Shepp 1977, Vershik–Kerov 1977). In 1993, Kerov announced a remarkable theorem describing Gaussian fluctuations around the limit shape  $\Omega$ . Here we propose a reconstruction of his proof. It is largely based on Kerov's unpublished work notes, 1999.

**Key words:** random Young diagrams, Plancherel measure, central limit theorem, generalized Gaussian processes

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## Contents

0. Introduction
1. The algebra of polynomial functions on the set of Young diagrams
2. Continual diagrams and their moments
3. The elements  $p_k^\#$
4. The basis  $\{p_\rho^\#\}$  and filtrations in  $\mathbb{A}$
5. The Plancherel measure and the law of large numbers
6. The central limit theorem for characters
7. The central limit theorem for Young diagrams
8. The central limit theorem for transition measures of Young diagrams
9. Discussion
10. Free cumulants and Biane's theorem

## 0. Introduction

### 0.1. MAIN RESULT

Let  $\mathbb{Y}_n$  denote the set of partitions of  $n$  ( $n = 1, 2, \dots$ ). We identify partitions and Young diagrams, so that elements of  $\mathbb{Y}_n$  become Young diagrams with  $n$  boxes. We view each  $\lambda \in \mathbb{Y}_n$  as a plane shape, of area  $n$ , inside the first quadrant  $\mathbb{R}_+^2$ , with coordinates  $r, s$  (the row and column coordinates). In new coordinates  $x = s - r$ ,  $y = r + s$ , the boundary  $\partial\lambda$  of the shape  $\lambda \subset \mathbb{R}_+^2$  may be viewed as the graph of a continuous piece-wise linear function, which we denote as  $y = \lambda(x)$ . Note that  $\lambda'(x) = \pm 1$ , and  $\lambda(x)$  coincides with  $|x|$  for sufficiently large values of  $|x|$ . The area of the shape  $|x| \leq y \leq \lambda(x)$  equals  $2n$ .

Further, we equip the finite set  $\mathbb{Y}_n$  with a probability measure  $M_n$  called the *Plancherel measure*. The measure  $M_n$  has important representation theoretic and combinatorial interpretations. By definition, the weight  $M_n(\lambda)$  assigned to a diagram  $\lambda \in \mathbb{Y}_n$  equals  $\dim^2 \lambda / n!$ , where  $\dim \lambda$  is the dimension of the irreducible representation (of the symmetric group  $\mathfrak{S}_n$ ) indexed by  $\lambda$ . Equivalently,  $\dim \lambda$  is the number of standard tableaux of shape  $\lambda$ .

Viewing  $\lambda$ 's as points of the probability space  $(\mathbb{Y}_n, M_n)$ , we view  $\lambda(\cdot)$ 's as *random functions*, and we aim to describe their asymptotics as  $n \rightarrow \infty$ . Informally, the main result can be stated as follows:

$$\frac{1}{\sqrt{n}} \lambda(\sqrt{n}x) \sim \Omega(x) + \frac{2}{\sqrt{n}} \Delta(x), \quad n \rightarrow \infty, \quad (1)$$

where  $\Omega(x)$  is a certain fixed curve and  $\Delta(x)$  is a generalized Gaussian process on the interval  $[-2, 2]$  ( $\Omega$  and  $\Delta$  are specified below).

The left-hand side (denoted as  $\bar{\lambda}(x)$  in the sequel) is a *rescaled version* of the function  $y = \lambda(x)$ . The graph of  $\bar{\lambda}(\cdot)$  is obtained from that of  $\lambda(\cdot)$  by shrinking both the  $x$ -axis and the  $y$ -axis in  $\sqrt{n}$  times. The purpose of this procedure is to put the random ensembles with different  $n$ 's on the same scale (note that the area of the shape  $|x| \leq y \leq \bar{\lambda}(x)$  equals 2 for any  $n$ ).

The first term in the right-hand side of (1) corresponds to the *law of large numbers*. It follows from (1) that in the large  $n$  limit, the random scaled polygonal lines  $y = \bar{\lambda}(x)$  concentrate near the fixed curve  $y = \Omega(x)$ . In the initial scale, this means that, for large  $n$ , the ‘‘typical’’ functions  $y = \lambda(x)$  look like the function  $y = \sqrt{n}\Omega(\frac{1}{\sqrt{n}}x)$ .

The second term in the right-hand side of (1) governs the *fluctuations* of the random functions  $\bar{\lambda}$  around the curve  $\Omega$ , which corresponds to the *central limit theorem*. We see that the fluctuations of the scaled functions are of order  $\frac{1}{\sqrt{n}}$ . That is, in the initial picture for the shape  $\lambda \subset \mathbb{R}_+^2$ , the random fluctuations of the boundary line  $\partial\lambda$  need no scaling *along the main diagonal*, as  $n \rightarrow \infty$ .

## 0.2. DESCRIPTION OF $\Omega$ AND $\Delta$

The function  $\Omega(x)$  is given by two different expressions depending on whether  $x$  is in the interval  $[-2, 2]$ :

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(x \arcsin \frac{x}{2} + \sqrt{4 - x^2}), & |x| \leq 2, \\ |x|, & |x| \geq 2. \end{cases} \quad (2)$$

Note that  $\Omega'(x) = \frac{2}{\pi} \arcsin \frac{x}{2}$  inside  $[-2, 2]$ . The critical points  $\pm 2$  have an important meaning: for ‘‘typical’’ (with respect to the Plancherel measure) diagrams  $\lambda \in \mathbb{Y}_n$ , the length of the first row and of the first column is approximately  $2\sqrt{n}$ . This claim, which seems plausible from (1) and (2) can be substantially refined, see [2], [1].

The Gaussian process  $\Delta(x)$  can be defined by a random trigonometric series. Let  $\xi_2, \xi_3, \dots$  be independent standard real Gaussian random variables (each  $\xi_k$  has mean 0 and variance 1), and set  $x = 2 \cos \theta$ , where  $0 \leq \theta \leq \pi$ . Then

$$\Delta(x) = \Delta(2 \cos \theta) = \frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\xi_k}{\sqrt{k}} \sin(k\theta). \quad (3)$$

For any smooth test function  $\varphi$  on  $\mathbb{R}$ , the smoothed series

$$\frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\xi_k}{\sqrt{k}} \int_{-2}^2 \sin(k\theta) \varphi(x) dx, \quad \theta = \arccos(x/2) \quad (4)$$

converges and is a Gaussian random variable. In this way we get a Gaussian measure on the space of distributions with support on  $[-2, 2]$ , or a generalized Gaussian process. Its trajectories are not ordinary functions but generalized functions.

### 0.3. HISTORY OF THE RESULT

The law of large numbers (the concentration near the curve  $\Omega$ ) was independently obtained by Logan and Shepp [27] and by Vershik and Kerov [38]. Their papers appeared in 1977. Later, in 1985, Vershik and Kerov published a detailed version of their work, [40], containing stronger results. In [27] and [40], the question about the second term of the asymptotics, corresponding to the central limit theorem, was posed.

Such a theorem was obtained by Kerov and announced in his short note [21], 1993. There Kerov also outlined the scheme of the proof. The note [21] contained a number of fruitful ideas, one of which (introduction of “good” coordinates in the set of Young diagrams) was largely developed in the joint note by Kerov and Olshanski [25], 1994.

For an intermediate result of [21], which is of independent interest, an elegant proof was suggested by Hora [14], 1998. Note that Hora’s approach differs from that of Kerov.

A few years ago we started to persuade Kerov to write a detailed exposition of his central limit theorem. Our discussions resulted first in the joint paper by Ivanov and Kerov [15], 1999, which clarified and developed one of the steps of Kerov’s proof.<sup>1</sup>

Then Kerov found a simpler derivation of the theorem, which also made apparent that the subject is connected with the concept of free cumulants and a theorem due to Biane [4]. In the end of 1999 Kerov sent us two short work notes with a description of the new approach. About the same time he gave a talk on this subject at Vershik’s seminar in St. Petersburg. He also started writing a detailed paper on this subject but had time only to finish the preliminary section.

In the present paper we give a detailed exposition of Kerov’s central limit theorem. Our aim was to reconstruct from his notes the “new approach”

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<sup>1</sup> It concerns the stable structure constants for convolution of conjugacy classes in symmetric groups. This topic was also discussed in [25].

of 1999.<sup>2</sup> This was not easy: for a long time we could not understand the meaning of some claims stated too briefly, but finally the picture became clear. However, we cannot be sure that we succeeded to completely fathom Kerov's intention, and there is no doubt that his own exposition would be quite different.

#### 0.4. LINKS WITH RANDOM MATRICES

Recently it was discovered that the limit distribution of a finitely many (properly scaled) first rows of the random Plancherel Young diagram  $\lambda \in \mathbb{Y}_n$ , as  $n \rightarrow \infty$ , coincides with the limit distribution of the same number of (properly scaled) largest eigenvalues of the random Hermitian  $N \times N$  matrix taken from the Gaussian Unitary Ensemble, as  $N \rightarrow \infty$ . See [2] and subsequent papers [29], [6], [18], [19], [3]. It turns out that the striking similarity between these two random ensembles holds not only "at the edge" (as is shown in these works) but also on the level of global fluctuations, which is the subject of the present paper. For spectra of random matrices, the limit behavior of global fluctuations was first studied in [8], where a central limit theorem was obtained. Further results in this direction were obtained in [16], [17], [7]. The generalized Gaussian processes that emerge in these works are very close to our process, we discuss this topic in Section 9.

#### 0.5. TECHNIQUES

Although the main result is stated in probabilistic terms, the techniques of the paper are essentially algebraic and combinatorial, the probabilistic part being reduced to a few elementary facts. The work is based on the choice of convenient "coordinate systems" for Young diagrams (there are several ones) and on the choice of an appropriate algebra  $\mathbb{A}$  of "observables". Elements of  $\mathbb{A}$  are functions on the set  $\mathbb{Y}$  of all Young diagrams. They are given by polynomial expressions in each of the "coordinate systems". For this reason we call  $\mathbb{A}$  the algebra of polynomial functions on  $\mathbb{Y}$ . We examine several different bases in  $\mathbb{A}$ . One of them (denoted as  $\{p_\rho^\#\}$ ) is related to the character table of the symmetric groups; this basis is well adapted to evaluating expectations with respect to the Plancherel measures  $M_n$ . Another basis has geometric significance; this basis is formed by monomials in  $\tilde{p}_2, \tilde{p}_3, \dots$ , a system of generators of  $\mathbb{A}$ , which are essentially the moments of  $\lambda(x)$ . One more basis (formed by monomials in generators  $p_1, p_2, \dots \in \mathbb{A}$ ) plays an intermediate role. A major part of our work consists in studying the transitions between various bases. This finally makes it possible to

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<sup>2</sup> It should be pointed out that the "old approach" of 1993 is also correct: we were able to directly check all the claims of [21].

isolate a good system of generators in  $\mathbb{A}$  that directly describe the Gaussian fluctuations.

## 0.6. ORGANIZATION OF THE PAPER

In Section 1, we introduce the algebra  $\mathbb{A}$  and a system  $p_1, p_2, \dots$  of its generators. We show that elements of  $\mathbb{A}$  are both shifted symmetric functions in the row coordinates  $\lambda_1, \lambda_2, \dots$  of a Young diagram  $\lambda \in \mathbb{Y}$ , and supersymmetric functions in the (modified) Frobenius coordinates of  $\lambda$ . This fact was first pointed out in [25].

In Section 2, we introduce the necessary geometric setting for visualizing fluctuations of Young diagrams. We embed  $\mathbb{Y}_n$  into the larger set  $\mathcal{D}^0$  of “continual diagrams”. We introduce the generators  $\tilde{p}_2, \tilde{p}_3, \dots \in \mathbb{A}$  and the “weight grading” of the algebra  $\mathbb{A}$ , which is well adapted to the operation of rescaling diagrams.

In Section 3, we examine one more system of generators in  $\mathbb{A}$ , denoted as  $p_1^\#, p_2^\#, \dots$ . These are character values on cycles in symmetric groups. We study the transitions between all three systems of generators. Here our tools are a suitably elaborated classical formula (due to Frobenius) for the value of a symmetric group character on the  $k$ -cycle, and Lagrange’s inversion formula.

In Section 4, we introduce the basis  $\{p_\rho^\#\}$  in  $\mathbb{A}$  and study a family of filtrations in  $\mathbb{A}$ , which are defined in terms of this basis. Here we follow the paper [15]. We essentially need two different filtrations. Their purpose is to single out main terms of asymptotics in different regimes. One filtration is responsible for the “law of large numbers” while another serves the “central limit theorem”.

In Section 5, we start the study of the Plancherel measures  $M_n$ . We introduce the sequence of expectation functionals  $\langle \cdot \rangle_n$  on  $\mathbb{A}$  that corresponds to the sequence  $\{M_n\}$ , and we remark that  $\langle \cdot \rangle_n$  becomes very simple in the basis  $\{p_\rho^\#\}$ . Then we prove the main result of the section — the law of large numbers, or convergence to the curve  $\Omega$ . Although the central limit theorem, established in Section 7, contains the law of large numbers, we prefer to prove it independently, because this can be done in a rather simple way. It is interesting to compare our simple algebraic argument with the analytic approach of the pioneer works [27], [38], [40].

In Section 6, we examine the random variables  $p_2^{\#(n)}, p_3^{\#(n)}, \dots$ , where  $p_k^{\#(n)}$  stands for the restriction of the function  $p_k^\# \in \mathbb{A}$  to the finite probability space  $(\mathbb{Y}_n, M_n)$ . We show that, as  $n \rightarrow \infty$ , the variables  $p_k^{\#(n)}$ , suitably scaled, are asymptotically independent Gaussians. This result is the first

version of the central limit theorem. Its proof relies on the method of [15]. A different proof has been given by Hora [14].

In Section 7, we obtain our main result: a description of the Gaussian fluctuations around the limit curve  $\Omega$ . It is derived from the central limit theorem for the generators  $p_k^\#$  mentioned above. The proof is based on a formula that gives the highest term of the polynomial expressing  $p_k^\#$  through the (centered and scaled versions of) the generators  $\tilde{p}_j$ . Here “highest term” refers to an appropriate filtration of the algebra  $\mathbb{A}$ , which we call Kerov’s filtration.

In Section 8, we get one more version of the central limit theorem. According to Vershik–Kerov’s theory, to any Young diagram  $\lambda$  we attach a probability measure on  $\mathbb{R}$  (say,  $\mu_\lambda$ ), supported by a finite set. Viewing  $\lambda \in \mathbb{Y}_n$  as the random element of the probability space  $(\mathbb{Y}_n, M_n)$ , we turn  $\mu_\lambda$  into a random measure. For these random measures we prove an asymptotic formula similar to (1), where, instead of the limit curve  $\Omega$ , we have the semi-circle distribution, and  $\Delta(x)$  is replaced by another generalized Gaussian process. We do not know if Kerov was aware of this result. However, it perfectly fits in the philosophy of his works.

In Section 9, we give comments to the results of Sections 7–8 and compare them with the central limit theorem for random matrices.

In Section 10, we show that the highest terms of the elements  $p_\rho^\#$  in the “weight grading” are closely related to the free cumulants. As an application, we get a simple proof of Biane’s asymptotic formula for character values of large symmetric groups, [4].

## 0.7. ACKNOWLEDGMENT

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## 1. The algebra of polynomial functions on the set of Young diagrams

Recall first the basic definitions and notation related to partitions and Young diagrams, see [28].

A *partition* is an infinite sequence  $\lambda = (\lambda_1, \lambda_2, \dots)$  of nonnegative integers such that  $\lambda_1 \geq \lambda_2 \geq \dots$  and the number of nonzero  $\lambda_i$ ’s is finite. The sum  $\lambda_1 + \lambda_2 + \dots$  is denoted by  $|\lambda|$ , and usually we set  $|\lambda| = n$ .

As in [28], we assign to a partition a Young diagram, which is denoted by the same symbol. We identify partitions and Young diagrams, and we denote by  $\mathbb{Y}$  the set of all Young diagrams. The conjugation involution of  $\mathbb{Y}$  (transposition of rows and columns of a diagram) is denoted as  $\lambda \mapsto \lambda'$ .

There is another presentation of Young diagrams, the *Frobenius notation*. We shall use its modification due to Vershik and Kerov [39]:

$$\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d).$$

Here  $d = d(\lambda)$  is the length of the main diagonal in  $\lambda$ ,

$$d(\lambda) = \{i \mid \lambda_i \geq i\} = \{j \mid \lambda'_j \geq j\},$$

and

$$a_i = \lambda_i - i + \frac{1}{2}, \quad b_i = \lambda'_i - i + \frac{1}{2}, \quad i = 1, \dots, d(\lambda). \quad (5)$$

The numbers  $a_i, b_i$  are called the *modified Frobenius coordinates* of  $\lambda$ . Both  $a_1, \dots, a_d$  and  $b_1, \dots, b_d$  are strictly decreasing positive proper half-integers, i.e., numbers from  $\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$ , such that  $\sum(a_i + b_i) = |\lambda|$ . Note that the Vershik–Kerov definition (5) differs from the classical definition of the Frobenius coordinates, which does not involve one-halves (see [28], p. 3). However, these one-halves play an important role in what follows.

Note that the conjugation involution  $\lambda \mapsto \lambda'$  has a very simple description in terms of the Frobenius coordinates:

$$(a_1, \dots, a_d \mid b_1, \dots, b_d)' = (b_1, \dots, b_d \mid a_1, \dots, a_d). \quad (6)$$

One more useful presentation of Young diagrams, due to Kerov, will be given in Section 2.

Set

$$\mathbb{Z}' = \mathbb{Z} + \frac{1}{2} = \{\dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots\}, \quad \mathbb{Z}'_+ = \{\frac{1}{2}, \frac{3}{2}, \dots\}, \quad \mathbb{Z}'_- = \{\dots, -\frac{3}{2}, -\frac{1}{2}\}.$$

Given  $\lambda \in \mathbb{Y}$ , set

$$l_i = \lambda_i - i + \frac{1}{2} \in \mathbb{Z}', \quad i = 1, 2, \dots,$$

and note that  $l_1 > l_2 > \dots$ . We assign to  $\lambda$  the infinite subset  $\mathcal{L}(\lambda) = \{l_1, l_2, \dots\} \subset \mathbb{Z}'$ .

The following claim is a version of the classical Frobenius lemma, see [28], Ch. I, (1.7) and Example 1.15(a).

**Proposition 1.1.** *Let  $\lambda \in \mathbb{Y}$  be arbitrary.*

(i) *We have  $\mathbb{Z}' = \mathcal{L}(\lambda) \sqcup (-\mathcal{L}(\lambda'))$ . I.e.,  $\mathcal{L}(\lambda) \cap (-\mathcal{L}(\lambda')) = \emptyset$  and  $\mathcal{L}(\lambda) \cup (-\mathcal{L}(\lambda')) = \mathbb{Z}'$ .*

(ii) *In the notation (5),*

$$\mathcal{L}(\lambda) \cap \mathbb{Z}'_+ = \{a_1, \dots, a_d\}, \quad \mathbb{Z}'_- \setminus \mathcal{L}(\lambda) = -(\mathcal{L}(\lambda') \cap \mathbb{Z}'_+) = \{-b_1, \dots, -b_d\}.$$

*Proof.* (i) We represent  $\lambda$  as a plane shape in the quarter plane  $\mathbb{R}_+^2$ . Let  $(r, s)$  be the coordinates in  $\mathbb{R}_+^2$ . Here the rows of  $\lambda$  are counted along the first coordinate  $r$ , directed downwards, while the columns are counted along the second coordinate  $s$ , directed to the right. Denote by  $\partial\lambda$  the doubly infinite polygonal line which first goes upwards along the  $r$ -axis, next goes along the boundary line separating  $\lambda$  from its complement in  $\mathbb{R}_+^2$ , and then goes to the right along the  $s$ -axis. For any  $a \in \mathbb{Z}'$ , the diagonal line  $s - r = a$  intersects  $\partial\lambda$  at the midpoint of a certain segment, which is either vertical or horizontal. According to these two possibilities  $a$  is either in  $\mathcal{L}(\lambda)$  or in  $-\mathcal{L}(\lambda')$ . This proves (i).

(ii) By the very definition (5), the numbers  $a \in \mathbb{Z}'_+$  such that the diagonal  $s - r = a$  meets a vertical boundary segment are exactly the numbers  $a_1, \dots, a_d$ . Likewise, the numbers  $-b \in \mathbb{Z}'_-$  such that the diagonal  $s - r = -b$  meets a horizontal boundary segment are exactly the numbers  $-b_1, \dots, -b_d$ . This proves (ii).

For any  $\lambda \in \mathbb{Y}$  we set

$$\Phi(z; \lambda) = \prod_{i=1}^{\infty} \frac{z + i - \frac{1}{2}}{z - \lambda_i + i - \frac{1}{2}}, \quad z \in \mathbb{C}.$$

The product is actually finite, because  $\lambda_i = 0$  when  $i$  is large enough. Therefore,  $\Phi(z; \lambda)$  is a rational function in  $z$ . We view it as a generating function of  $\lambda$ .

**Proposition 1.2.** *In the notation (5), we have*

$$\Phi(z; \lambda) = \prod_{i=1}^d \frac{z + b_i}{z - a_i}, \quad (7)$$

*which is the presentation of  $\Phi(z; \lambda)$  as an incontractible fraction.*

*Proof.* The equality (7) follows from Proposition 1.1. This is an incontractible fraction, because the numbers  $a_1, \dots, a_d, -b_1, \dots, -b_d$  are pairwise distinct.

Another proof of the equality (7) is given in [31], it follows an idea from [25].

As the first corollary of (7) note the relation

$$\Phi(z; \lambda') = 1/\Phi(-z; \lambda).$$

Remark that  $\Phi(z; \lambda) = 1 + O(\frac{1}{z})$  near  $z = \infty$ , hence both  $\Phi(z; \lambda)$  and  $\ln \Phi(z; \lambda)$  can be expanded in a power series in  $z^{-1}, z^{-2}, \dots$  about  $z = \infty$ .

**Definition 1.3.** *The algebra of polynomial functions on the set  $\mathbb{Y}$ , denoted as  $\mathbb{A}$ , is generated over  $\mathbb{R}$  by the coefficients of the above expansion of  $\Phi(z; \lambda)$  or, equivalently, of  $\ln \Phi(z; \lambda)$ . We also assume that  $\mathbb{A}$  contains 1.*

**Proposition 1.4.** *We have*

$$\ln \Phi(z; \lambda) = \sum_{k=1}^{\infty} \frac{p_k(\lambda)}{k} z^{-k},$$

where

$$p_k(\lambda) = \sum_{i=1}^{\infty} [(\lambda_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k] \quad (8)$$

$$= \sum_{i=1}^{d(\lambda)} [a_i^k - (-b_i)^k]. \quad (9)$$

*Proof.* Immediate from Proposition 1.2.

Thus,  $\mathbb{A}$  is generated by the functions  $p_k(\lambda)$ ,  $k = 1, 2, \dots$

Recall [28] that the *algebra of symmetric functions*, denoted as  $\Lambda$ , is the graded algebra defined as the projective limit (in the category of graded algebras)  $\varprojlim \Lambda(n)$ , where  $\Lambda(n)$  denotes the algebra of symmetric polynomials in  $n$  variables. As the base field we take  $\mathbb{R}$ . The morphism  $\Lambda(n) \rightarrow \Lambda(n-1)$ , which is employed in the projective limit transition, is defined as specializing the  $n$ th variable to 0. Let  $\{\mathbf{h}_k\}_{k=1,2,\dots}$  and  $\{\mathbf{p}_k\}_{k=1,2,\dots}$  denote the complete homogeneous symmetric functions and the Newton power sums, respectively. Each of these two families is a system of homogeneous, algebraically independent generators of  $\Lambda$ ,  $\deg \mathbf{h}_k = \deg \mathbf{p}_k = k$ . Recall the basic relation:

$$1 + \sum_{k=1}^{\infty} \mathbf{h}_k t^k = \exp \sum_{k=1}^{\infty} \frac{\mathbf{p}_k}{k} t^k.$$

**Proposition 1.5.** *The generators  $p_k \in \mathbb{A}$  are algebraically independent, so that  $\mathbb{A}$  is isomorphic to  $\mathbb{R}[p_1, p_2, \dots]$ .*

*Proof.* Fix an arbitrary  $N = 1, 2, \dots$ . Assume that  $f$  is a polynomial in  $N$  variables such that  $f(p_1, \dots, p_N) = 0$ , and show that  $f = 0$ . Let  $\bar{f}$  denote the top homogeneous component of  $f$  counted with the understanding that the degree of the  $i$ th variable equals  $i$ ; it suffices to show that  $\bar{f} = 0$ .

Let  $\lambda$  range over the set of partitions of length  $\leq N$ . Fix an arbitrary vector  $x \in \mathbb{R}^N$  with nonnegative weakly decreasing coordinates and set  $\lambda = \lambda(A) = ([Ax]_i)_{i=1,\dots,N}$ , where  $A$  is a large integer. Letting  $A \rightarrow \infty$  in

the equality  $f(p_1(\lambda(A)), \dots, p_N(\lambda(A))) = 0$  we get  $\bar{f}(\mathbf{p}_1(x), \dots, \mathbf{p}_N(x)) = 0$ . Since the first  $N$  Newton power sums specialized in  $N$  variables are algebraically independent, we conclude that  $\bar{f} = 0$ .

**Definition 1.6.** *Setting  $\Lambda \ni \mathbf{p}_k \mapsto p_k \in \mathbb{A}$  and taking into account Proposition 1.5 we get an algebra isomorphism  $\Lambda \rightarrow \mathbb{A}$ . We call it the canonical isomorphism. We call the grading in  $\mathbb{A}$ , inherited from that of  $\Lambda$ , the canonical grading of  $\mathbb{A}$ .*

Later on, in Definition 2.9, we shall define quite a different grading in  $\mathbb{A}$ .

In terms of generating series, the canonical isomorphism  $\Lambda \rightarrow \mathbb{A}$  takes the form

$$H(t) := 1 + \sum_{k=1}^{\infty} \mathbf{h}_k t^k \mapsto \Phi(t^{-1}; \cdot).$$

Formula (9) means that the functions  $p_k(\lambda)$  are *super* power sums in  $a_i$ 's and  $b_i$ 's, see [28], Example I.3.23, [39], [25], [31]. Thus, one can say that under the canonical isomorphism of Definition 1.6, the algebra  $\mathbb{A}$  is identified with the *algebra of supersymmetric functions in the modified Frobenius coordinates* of a Young diagram.

Next, we shall give a similar interpretation of formula (8). Recall [30] that the *algebra of shifted symmetric functions*, denoted as  $\Lambda^*$ , is the filtered algebra defined as the projective limit (in the category of filtered algebras)  $\varprojlim \Lambda^*(n)$ , where  $\Lambda^*(n)$  consists of those polynomials in  $n$  variables  $x_1, \dots, x_n$ , which become symmetric in new variables  $y_i = x_i - i + \text{const}$  (the choice of the constant here is irrelevant). The base field is again  $\mathbb{R}$ , the filtration is taken with respect to the total degree of a polynomial, and the morphism  $\Lambda^*(n) \rightarrow \Lambda^*(n-1)$  is defined as above, i.e., as specializing  $x_n = 0$ . The graded algebra associated to the filtered algebra  $\Lambda^*$  is canonically isomorphic to  $\Lambda$ . The algebra  $\Lambda^*$  is generated by the algebraically independent system  $\{\mathbf{p}_k^*\}_{k=1,2,\dots}$ , where

$$\mathbf{p}_k^*(x_1, x_2, \dots) = \sum_{i=1}^{\infty} [(x_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k], \quad k = 1, 2, \dots,$$

are certain shifted analogs of the Newton power sums. See [30] for more detail (note that the above definition of the elements  $\mathbf{p}_k^*$  slightly differs from that given in [30]). See also [10].

By analogy with Definition 1.6, we define an algebra isomorphism  $\Lambda^* \rightarrow \mathbb{A}$  by setting  $\mathbf{p}_k^* \mapsto p_k$ ,  $k = 1, 2, \dots$ . Note that it preserves the filtration. Then formula (8) makes it possible to say that the algebra  $\mathbb{A}$  coincides with the *algebra of shifted symmetric functions in the row coordinates*  $\lambda_1, \lambda_2, \dots$  of a Young diagram  $\lambda$ .

**Definition 1.7.** Define an involutive algebra automorphism  $\text{inv} : \mathbb{A} \rightarrow \mathbb{A}$  by

$$(\text{inv}(f))(\lambda) = f(\lambda'), \quad f \in \mathbb{A}, \quad \lambda \in \mathbb{Y}. \quad (10)$$

By virtue of (6) and (9),

$$\text{inv}(p_k) = (-1)^{k-1} p_k, \quad k = 1, 2, \dots \quad (11)$$

Hence the involution of  $\mathbb{A}$  is compatible with the canonical involution of the algebra  $\Lambda$  with respect to the isomorphism  $\Lambda \rightarrow \mathbb{A}$  introduced in Definition 1.6.

## 2. Continual diagrams and their moments

**Definition 2.1.** A continual diagram is a function  $\omega(x)$  on  $\mathbb{R}$  such that:

(i)  $|\omega(x_1) - \omega(x_2)| \leq |x_1 - x_2|$  for any  $x_1, x_2 \in \mathbb{R}$  (the Lipschitz condition).

(ii) There exists a point  $x_0 \in \mathbb{R}$ , called the center of  $\omega$ , such that  $\omega(x) = |x - x_0|$  when  $|x|$  is large enough.

The set of all continual diagrams is denoted by  $\mathcal{D}$ , and the subset of diagrams with center 0 is denoted by  $\mathcal{D}^0$ .

This definition is due to Kerov (see his papers [22], [23], [24]). We shall mainly deal with the set  $\mathcal{D}^0$ .

To any  $\omega \in \mathcal{D}$  we assign a function  $\sigma(x)$ :

$$\sigma(x) = \frac{1}{2}(\omega(x) - |x|). \quad (12)$$

Since  $\sigma(x)$  satisfies the Lipschitz condition (i), its derivative  $\sigma'(x)$  exists almost everywhere and satisfies  $|\sigma'(x)| \leq 1$ . By (ii), the function  $\sigma'(x)$  is compactly supported.

If  $\omega \in \mathcal{D}^0$  then  $\sigma(x)$  is compactly supported, too. For general  $\omega \in \mathcal{D}$ , we have

$$\sigma(x) \equiv -x_0, \quad x \gg 0; \quad \sigma(x) \equiv x_0, \quad x \ll 0.$$

This implies that  $\omega(x)$  is uniquely determined by  $\sigma'(x)$ . Even more,  $\omega(x)$  is uniquely determined by the second derivative  $\sigma''(x)$ , which is understood in the sense of distribution theory.

Define the functions  $\tilde{p}_1, \tilde{p}_2, \dots$  on  $\mathcal{D}$  by setting

$$\tilde{p}_k[\omega] = -k \int_{-\infty}^{\infty} x^{k-1} \sigma'(x) dx \quad (13)$$

$$= \int_{-\infty}^{\infty} x^k \sigma''(x) dx, \quad (14)$$

where  $\omega \in \mathcal{D}$ ,  $k = 1, 2, \dots$

**Proposition 2.2.** *If  $\omega \in \mathcal{D}^0$  then  $\tilde{p}_1[\omega] = 0$  and*

$$\tilde{p}_k[\omega] = k(k-1) \int_{-\infty}^{\infty} x^{k-2} \sigma(x) dx, \quad k = 2, 3, \dots$$

*Proof.* Recall that  $\sigma(x)$  is finitely supported when  $\omega \in \mathcal{D}^0$ . This implies the first claim. Further, integrating (13) by parts gives the second claim.

**Definition 2.3.** *Given  $\lambda \in \mathbb{Y}$ , we define a piece-wise linear function  $\lambda(\cdot)$  as follows. Let  $(r, s)$  and  $\partial\lambda$  be as in the proof of Proposition 1.1. Then the graph  $y = \lambda(x)$  describes  $\partial\lambda$  in the coordinates  $x = s - r$ ,  $y = r + s$ . The correspondence  $\lambda \mapsto \lambda(\cdot)$  yields an embedding  $\mathbb{Y} \hookrightarrow \mathcal{D}^0$ .*

We have  $\lambda'(x) = \pm 1$ , except finitely many points, which are exactly the local extrema of the function  $\lambda(x)$ . These local extrema form two interlacing sequences of points

$$x_1 < y_1 < x_2 < \dots < x_m < y_m < x_{m+1}, \quad (15)$$

where the  $x_i$ 's are the local minima and the  $y_j$ 's are the local maxima of the function  $\lambda(x)$ .

**Proposition 2.4.** *We have*

$$x_i \in \mathbb{Z}, \quad y_j \in \mathbb{Z}, \quad \sum x_i - \sum y_j = 0. \quad (16)$$

*Conversely, any couple of interlacing sequences (15) satisfying (16) comes from a Young diagram  $\lambda$ , which is determined uniquely.*

*Idea of proof.* For any couple of interlacing sequences (15), there exists a unique polygonal line  $\omega \in \mathcal{D}$ , with center at  $x_0 = \sum x_i - \sum y_j$  and such that, for the corresponding function  $\sigma$ ,

$$\sigma''(x) = \sum_{i=1}^{m+1} \delta(x - x_i) - \sum_{j=1}^m \delta(x - y_j) - \delta(x - x_0).$$

The line  $\omega$  represents a Young diagram if and only if  $x_0 = 0$ . □

The correspondence  $\lambda \mapsto \{x_i\} \cup \{y_j\}$  provides one more useful system of parameters for Young diagrams.

**Proposition 2.5.** *Let  $\lambda \in \mathbb{Y}$ , let  $\lambda(\cdot) \in \mathcal{D}^0$  be the corresponding continual diagram, and consider the local extrema (15). We have*

$$\tilde{p}_k[\lambda(\cdot)] = \sum_{i=1}^{m+1} x_i^k - \sum_{j=1}^m y_j^k, \quad k = 1, 2, \dots$$

*Proof.* Let  $\sigma$  be associated with  $\omega = \lambda(\cdot)$ , as defined in (12). Then we get

$$\sigma''(x) = \sum_{i=1}^{m+1} \delta(x - x_i) - \sum_{j=1}^m \delta(x - y_j) - \delta(x).$$

Note that  $\int x^k \delta(x) dx = 0$  for any  $k = 1, 2, \dots$  and apply (14).

**Proposition 2.6.** *Let  $\lambda \in \mathbb{Y}$  and let  $\{x_i\} \cup \{y_j\}$  be the local extrema of  $\lambda(\cdot)$ . The following identity holds*

$$\frac{\Phi(z - \frac{1}{2}; \lambda)}{\Phi(z + \frac{1}{2}; \lambda)} = \frac{z \prod_{j=1}^m (z - y_j)}{\prod_{i=1}^{m+1} (z - x_i)}. \quad (17)$$

*Proof.* We shall prove the identity

$$\Phi(z - \frac{1}{2}; \lambda) = \frac{\prod_{i=1}^{m+1} \Gamma(z - x_i)}{\Gamma(z) \prod_{j=1}^m \Gamma(z - y_j)}, \quad (18)$$

which implies (17). Using (7), we rewrite (18) as

$$\prod_{i=1}^d \frac{z + b_i}{z - a_i} = \frac{\prod_{i=1}^{m+1} \Gamma(z - x_i + \frac{1}{2})}{\Gamma(z + \frac{1}{2}) \prod_{j=1}^m \Gamma(z - y_j + \frac{1}{2})}. \quad (19)$$

Let  $(r, s)$  be the row and column coordinates in the quarter-plane, see the proof of Proposition 1.1. Draw the diagonal lines  $s - r = x_i$  ( $1 \leq i \leq m+1$ ) and  $s - r = y_j$  ( $1 \leq j \leq m$ ), which divide the boundary line  $\partial\lambda$  into interlacing vertical and horizontal pieces.

Assume first that  $x_l < 0 < y_l$  for a certain  $l$ . Consider an arbitrary *vertical* piece of  $\partial\lambda$  which sits *above* the main diagonal  $s - r = 0$ . The ends of such a piece lie on the lines  $s - r = y_k$  and  $s - r = x_{k+1}$ , where  $l \leq k \leq m$ . Inside this piece, the row Frobenius coordinates increase by one and form the sequence

$$y_k + \frac{1}{2}, \quad y_k + \frac{3}{2}, \quad \dots, \quad x_{k+1} - \frac{3}{2}, \quad x_{k+1} - \frac{1}{2},$$

so that the partial product in  $\prod_{i=1}^d 1/(z - a_i)$  corresponding to this sequence equals

$$\frac{\Gamma(z - x_{k+1} + \frac{1}{2})}{\Gamma(z - y_k + \frac{1}{2})}.$$

It follows that

$$\prod_{i=1}^d \frac{1}{z - a_i} = \prod_{k=l}^m \frac{\Gamma(z - x_{k+1} + \frac{1}{2})}{\Gamma(z - y_k + \frac{1}{2})} = \frac{\prod_{k=l+1}^{m+1} \Gamma(z - x_k + \frac{1}{2})}{\prod_{k=l}^m \Gamma(z - y_k + \frac{1}{2})}. \quad (20)$$

Next, consider a *horizontal* piece *below* the main diagonal. Such a piece sits between the diagonal lines  $s-r = x_k$  and  $s-r = y_k$ , where  $1 \leq k \leq l-1$ . In this piece, the column Frobenius coordinates make up the sequence

$$-(x_k + \frac{1}{2}), \quad -(x_k + \frac{3}{2}), \quad \dots, \quad -(y_k - \frac{3}{2}), \quad -(y_k - \frac{1}{2}),$$

whose contribution to the product  $\prod_{i=1}^d (z + b_i)$  equals

$$\frac{\Gamma(z - x_k + \frac{1}{2})}{\Gamma(z - y_k + \frac{1}{2})}.$$

Therefore, the contribution of all horizontal pieces below the main diagonal equals

$$\prod_{k=1}^{l-1} \frac{\Gamma(z - x_k + \frac{1}{2})}{\Gamma(z - y_k + \frac{1}{2})}. \quad (21)$$

Finally, consider the only piece that *intersects* the main diagonal. By our assumption, this piece is *horizontal*, and it sits between the lines  $s-r = x_l$  and  $s-r = y_l$ . We have to examine the row Frobenius coordinates inside it. They make up the sequence

$$-(x_l + \frac{1}{2}), \quad -(x_l + \frac{3}{2}), \quad \dots, \quad \frac{1}{2}$$

(recall that  $x_l < 0 < y_l$ ). The corresponding contribution equals

$$\frac{\Gamma(z - x_l + \frac{1}{2})}{\Gamma(z + \frac{1}{2})}. \quad (22)$$

Multiplying up (21) and (22) we get

$$\begin{aligned} \prod_{i=1}^d (z + b_i) &= \prod_{k=1}^{l-1} \frac{\Gamma(z - x_k + \frac{1}{2})}{\Gamma(z - y_k + \frac{1}{2})} \cdot \frac{\Gamma(z - x_l + \frac{1}{2})}{\Gamma(z + \frac{1}{2})} \\ &= \frac{\prod_{k=1}^l \Gamma(z - x_k + \frac{1}{2})}{\Gamma(z + \frac{1}{2}) \prod_{k=1}^{l-1} \Gamma(z - y_k + \frac{1}{2})}. \end{aligned} \quad (23)$$

Now (19) follows from (20) and (23).

We have verified (17) under the assumption  $x_l < 0 < y_l$ , i.e., in the case when the main diagonal  $s-r = 0$  meets the boundary line  $\partial\lambda$  at an interior point of a horizontal piece. The same argument works if the intersection of  $s-r = 0$  with  $\partial\lambda$  is inside a vertical piece (i.e.,  $y_l < 0 < x_{l+1}$  for a certain  $l$ ) or if the intersection point coincides with a brake of  $\partial\lambda$  (i.e., some  $x_l$  or  $y_l$  is 0).

Set

$$\tilde{p}_k(\lambda) = \tilde{p}_k[\lambda(\cdot)], \quad k = 1, 2, \dots, \quad \lambda \in \mathbb{Y},$$

where the right-hand side is given by (13) or, equivalently, by (14). Note that  $\tilde{p}_1(\lambda) \equiv 0$ .

**Proposition 2.7.** *The functions  $\tilde{p}_2(\lambda), \tilde{p}_3(\lambda), \dots$  belong to the algebra  $\mathbb{A}$  and are related to the functions  $p_1(\lambda), p_2(\lambda)$  by the relations*

$$\tilde{p}_k = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} 2^{-2j} p_{k-1-2j}, \quad k = 2, 3, \dots \quad (24)$$

*Proof.* Formula (17) implies that

$$\ln \Phi(z - \frac{1}{2}; \lambda) - \ln \Phi(z + \frac{1}{2}; \lambda) = \ln \prod_{j=1}^m (1 - \frac{y_j}{z}) - \ln \prod_{i=1}^{m+1} (1 - \frac{x_i}{z}). \quad (25)$$

By Proposition 1.4, the left-hand side equals

$$\begin{aligned} & \sum_{l=1}^{\infty} \frac{p_l(\lambda)}{l} \left( \frac{1}{(z - \frac{1}{2})^l} - \frac{1}{(z + \frac{1}{2})^l} \right) \\ &= \sum_{l=1}^{\infty} \frac{p_l(\lambda)}{l} z^{-l} \left( \left(1 - \frac{1}{2z}\right)^{-l} - \left(1 + \frac{1}{2z}\right)^{-l} \right) \\ &= \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} \frac{l(l+1)\dots(l+2j)}{(2j+1)!} \frac{p_l(\lambda)}{l} \frac{z^{-(l+2j+1)}}{2^{2j}} \\ &= \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} \frac{(l+1)\dots(l+2j+1)}{(2j+1)!} \frac{p_l(\lambda)}{l+2j+1} \frac{z^{-(l+2j+1)}}{2^{2j}}. \end{aligned}$$

Setting  $l + 2j + 1 = k$  we rewrite this as

$$\sum_{k=2}^{\infty} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} \frac{p_{k-2j-1}(\lambda)}{2^{2j}} \frac{z^{-k}}{k}. \quad (26)$$

By Proposition 2.5, the right-hand side of (25) equals

$$\sum_{k=1}^{\infty} \tilde{p}_k(\lambda) \frac{z^{-k}}{k}. \quad (27)$$

Comparing the coefficients of  $z^{-k}/k$  in (26) and (27) we get  $\tilde{p}_1(\lambda) \equiv 0$  (which we already know) and then (24).

Note that

$$\text{inv}(\tilde{p}_k) = (-1)^k \tilde{p}_k, \quad k = 2, 3, \dots, \quad (28)$$

where ‘inv’ is the involution introduced in Definition 1.7. Indeed, (28) easily follows from the definition of  $\tilde{p}_k$  and the symmetry property  $\lambda'(x) = \lambda(-x)$ . The fact that in the right-hand side of (24), the subscript varies with step 2 agrees with the symmetry properties of  $p_k$ 's and  $\tilde{p}_k$ 's, see (11) and (28).

**Corollary 2.8.** *For any  $k = 2, 3, \dots$*

$$\frac{\tilde{p}_k}{k} = p_{k-1} + \langle \text{a linear combination of } p_{k-2}, \dots, p_1 \rangle.$$

*Conversely, for any  $k = 1, 2, \dots$*

$$p_k = \frac{\tilde{p}_{k+1}}{k+1} + \langle \text{a linear combination of } \tilde{p}_k, \dots, \tilde{p}_2 \rangle.$$

By Corollary 2.8, the elements  $\tilde{p}_2, \tilde{p}_3, \dots$  are algebraically independent generators of the algebra  $\mathbb{A}$ :

$$\mathbb{A} = \mathbb{R}[\tilde{p}_2, \tilde{p}_3, \dots].$$

**Definition 2.9.** (cf. [10]) *The weight grading of the algebra  $\mathbb{A}$  is defined by setting*

$$\text{wt}(\tilde{p}_k) = k, \quad k = 2, 3, \dots$$

*Equivalently, the weight grading is the image of the standard grading of  $\Lambda$  under the algebra morphism*

$$\begin{aligned} \Lambda = \mathbb{R}[\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \dots] &\rightarrow \mathbb{A} = \mathbb{R}[\tilde{p}_2, \tilde{p}_3, \dots], \\ \mathbf{p}_1 &\rightarrow 0, \quad \mathbf{p}_k \rightarrow \tilde{p}_k, \quad k = 2, 3, \dots \end{aligned} \quad (29)$$

This definition is motivated by Proposition 2.11 below.

This morphism induces an algebra isomorphism  $\Lambda/\mathbf{p}_1\Lambda \rightarrow \mathbb{A}$ . Let us emphasize the difference from the isomorphism  $\Lambda \rightarrow \mathbb{A}$  (Definition 1.6).

The weight grading induces a filtration in  $\mathbb{A}$ , which we call the *weight filtration* and denote by the same symbol  $\text{wt}(\cdot)$ . Note that

$$\text{wt}(p_k) = k + 1, \quad k = 1, 2, \dots,$$

because the top weight homogeneous component of  $p_k$  is  $\tilde{p}_{k+1}/(k+1)$ , see Corollary 2.8.

**Definition 2.10.** *a) We define an action of the multiplicative group of positive real numbers on the set  $\mathcal{D}^0$  by setting*

$$\omega^s(x) = s^{-1}\omega(sx), \quad \omega \in \mathcal{D}^0, \quad s > 0, \quad x \in \mathbb{R}.$$

In other words, the graph of  $y = \omega^s(x)$  is obtained from that of  $y = \omega(x)$  by the transformation  $(x, y) \mapsto (s^{-1}x, s^{-1}y)$ .

b) Since  $\mathbb{A} = \mathbb{R}[\tilde{p}_2, \tilde{p}_3, \dots]$ , we may define the symbol  $f[\omega]$  (where  $\omega$  ranges over  $\mathcal{D}^0$ ) for any  $f \in \mathbb{A}$ . Specifically, write  $f$  as a polynomial in  $\tilde{p}_2, \tilde{p}_3, \dots$  and then specialize each  $\tilde{p}_k$  to  $\tilde{p}_k[\omega]$ . In this way, we realize  $\mathbb{A}$  as an algebra of functions on  $\mathcal{D}^0$ .

**Proposition 2.11.** *Let  $f \in \mathbb{A}$  be homogeneous with respect to the weight grading, Definition 2.9. Then for any  $\omega \in \mathcal{D}^0$  and  $s > 0$ ,*

$$f[\omega^s] = s^{-\text{wt}(f)} f[\omega].$$

*Proof.* By the definition of the weight grading, it suffices to check that

$$\tilde{p}_k[\omega^s] = s^{-k} \tilde{p}_k[\omega], \quad k = 2, 3, \dots, \quad \omega \in \mathcal{D}^0, \quad s > 0. \quad (30)$$

Remark that the function  $\sigma(x) = \frac{1}{2}(\omega(x) - |x|)$  transforms in the same way as  $\omega(x)$ . Then (30) is clear from Proposition 2.2.

### 3. The elements $p_k^\#$

Let  $\mathfrak{S}_n$  be the symmetric group of degree  $n$ . Recall that both irreducible characters and conjugacy classes of  $\mathfrak{S}_n$  are indexed by the same set, the set of partitions of  $n$  or, equivalently, of Young diagrams with  $n$  boxes. We denote this set by  $\mathbb{Y}_n$ . For  $\lambda, \rho \in \mathbb{Y}_n$ , we denote by  $\chi^\lambda$  the irreducible character of  $\mathfrak{S}_n$  indexed by  $\lambda$ , and by  $\chi_\rho^\lambda$  the value of  $\chi^\lambda$  on the conjugacy class indexed by  $\rho$ .

In particular, the partition  $\rho = (1^n) = (1, \dots, 1)$  corresponds to the trivial conjugacy class  $\{e\} \subset \mathfrak{S}_n$ , so that  $\chi_{(1^n)}^\lambda$  equals the dimension of  $\chi^\lambda$ ; we denote this number by  $\dim \lambda$ .

**Definition 3.1.** *For  $k = 1, 2, \dots$ , let  $p_k^\#$  be following function on  $\mathbb{Y}$ :*

$$p_k^\#(\lambda) = \begin{cases} n^{\downarrow k} \cdot \frac{\chi_{(k, 1^{n-k})}^\lambda}{\dim \lambda}, & n := |\lambda| \geq k, \\ 0, & n < k, \end{cases}$$

where

$$n^{\downarrow k} = n(n-1) \dots (n-k+1)$$

and

$$(k, 1^{n-k}) = (k, 1, \dots, 1) \in \mathbb{Y}_n.$$

**Proposition 3.2.** For any  $k = 1, 2, \dots$  and any  $\lambda \in \mathbb{Y}$ ,  $p_k^\#(\lambda)$  equals the coefficient of  $z^{-1}$  in the expansion of the function

$$-\frac{1}{k} \left(z - \frac{1}{2}\right) \downarrow k \frac{\Phi(z; \lambda)}{\Phi(z - k; \lambda)} \quad (31)$$

in descending powers of  $z$  about the point  $z = \infty$ .

*Proof.* First, assume  $n < k$ . Then, by the definition,  $p_k^\#(\lambda) = 0$ , and we have to prove that the coefficient in question equals 0, too. It suffices to prove that (31) is a polynomial in  $z$ . By Proposition 1.2, (31) equals

$$-\frac{1}{k} \left(z - \frac{1}{2}\right) \downarrow k \prod_{i=1}^d \frac{z + b_i}{z - a_i} \cdot \prod_{j=1}^d \frac{z - a_j - k}{z + b_j - k}.$$

Note that  $a_i \neq k - b_j$  for any  $i, j = 1, \dots, d$ , because  $a_i + b_j \leq n < k$ . Therefore, all the factors  $z - a_i$  and  $z + b_j - k = z - (k - b_j)$  are pairwise distinct. Each of them cancels with one of the factors in the product  $(z - \frac{1}{2}) \downarrow k$ , because

$$a_i, k - b_j \in \left\{\frac{1}{2}, \frac{3}{2}, \dots, k - \frac{1}{2}\right\}.$$

This concludes the proof in the case  $n < k$ .

Now we shall assume  $n \geq k$ . Then we use a formula due to Frobenius (see [28], Ex. I.7.7) which says that  $p_k^\#(\lambda)$  equals the coefficient of  $z^{-1}$  in the expansion of the function

$$F(z) = -\frac{1}{k} z \downarrow k \prod_{i=1}^n \frac{z - \lambda_i - n + i - k}{z - \lambda_i - n + i}$$

about  $z = \infty$ . In other words,

$$p_k^\#(\lambda) = -\text{Res}_{z=\infty}(F(z)dz).$$

After simple transformations we get

$$F(z) = -\frac{1}{k} (z - n) \downarrow k \frac{\Phi(z - n + \frac{1}{2}; \lambda)}{\Phi(z - n + \frac{1}{2} - k; \lambda)}.$$

The residue at  $z = \infty$  will not change under the shift  $z \mapsto z + n - \frac{1}{2}$ . Consequently,

$$p_k^\#(\lambda) = -\text{Res}_{z=\infty} \left( -\frac{1}{k} \left(z - \frac{1}{2}\right) \downarrow k \frac{\Phi(z; \lambda)}{\Phi(z - k; \lambda)} \right),$$

which completes the proof.

We shall employ the following notation. Given a formal series  $A(t)$ , let

$$[t^k]\{A(t)\} = \langle \text{the coefficient of } t^k \text{ in } A(t) \rangle.$$

The next result is due to Wassermann [43], §III.6.

**Proposition 3.3.** *For any  $k = 1, 2, \dots$ , the function  $p_k^\#(\lambda)$  introduced in Definition 2.1 belongs to the algebra  $\mathbb{A}$ . Its expression through the generators  $p_1, p_2, \dots$  of  $\mathbb{A}$  can be described as follows:*

$$p_k^\# = [t^{k+1}] \left\{ -\frac{1}{k} \prod_{j=1}^k (1 - (j - \frac{1}{2})t) \cdot \exp \left( \sum_{j=1}^{\infty} \frac{p_j t^j}{j} (1 - (1 - kt)^{-j}) \right) \right\}. \quad (32)$$

*Proof.* By Proposition 3.2,

$$p_k^\#(\lambda) = [t^{k+1}] \left\{ -\frac{1}{k} t^k (t^{-1} - \frac{1}{2})^{\downarrow k} \frac{\Phi(t^{-1}; \lambda)}{\Phi \left( \left( \frac{t}{1-kt} \right)^{-1}; \lambda \right)} \right\}.$$

We have

$$t^k (t^{-1} - \frac{1}{2})^{\downarrow k} = \prod_{j=1}^k (1 - (j - \frac{1}{2})t)$$

and, by Proposition 1.4,

$$\Phi(t^{-1}; \lambda) = \exp \left( \sum_{j=1}^{\infty} \frac{p_j(\lambda)}{j} t^j \right).$$

This yields (32), which in turn implies that  $p_k^\# \in \mathbb{A}$ .

The expression (32) can be written in the form

$$\begin{aligned} p_k^\# &= -\frac{1}{k} [t^{k+1}] \left\{ (1 + \varepsilon_0(t)) \exp \left( -\sum_{j=1}^{\infty} k p_j t^{j+1} (1 + \varepsilon_j(t)) \right) \right\} \\ &= -\frac{1}{k} [t^{k+1}] \left\{ (1 + \varepsilon_0(t)) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left( \sum_{j=1}^{\infty} k p_j t^{j+1} (1 + \varepsilon_j(t)) \right)^m \right\}. \quad (33) \end{aligned}$$

Here each  $\varepsilon_r(t)$  is a power series of the form  $c_1 t + c_2 t^2 + \dots$ , where the coefficients  $c_1, c_2, \dots$  do not involve the generators  $p_1, p_2, \dots$ .

Using (33) we can readily evaluate the top homogeneous component of  $p_k^\#$  with respect both to the canonical grading and the weight grading in  $\mathbb{A}$ .

**Proposition 3.4.** *In the canonical grading, see Definition 1.6, the highest term of  $p_k^\#$  equals  $p_k$ .*

*Proof.* Apply (33) and write the expression in the curly brackets as a sum of terms of the form  $\text{const} \cdot P \cdot t^r$ , where  $P$  stands for a monomial in  $p_1, p_2, \dots$ . We search for terms with  $r = k + 1$  and such that  $\deg P$ , the total degree of  $P$ , counted with the convention that  $\deg p_k = k$ , is maximal possible.

The first observation is that all terms  $\text{const} \cdot P \cdot t^r$  involving at least one factor coming from  $\varepsilon_0(t), \varepsilon_1(t), \dots$  are negligible, because the epsilon factors diminish the difference  $\deg P - r$ . Removing  $\varepsilon_0(t), \varepsilon_1(t), \dots$ , we get

$$p_k^\# = -\frac{1}{k} [t^{k+1}] \left\{ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left( \sum_{j=1}^{\infty} k p_j t^{j+1} \right)^m \right\} + \dots,$$

where dots stand for lower degree terms. The summand with  $m = 1$  has a unique term with  $r = k + 1$ . This term is  $-k p_k t^{k+1}$ , and its contribution is  $p_k$ .

The second observation is that the summands with  $m = 2, 3, \dots$  are negligible, because, in the corresponding terms,  $\deg P - r = -m \leq -2$ , so that  $r = k + 1$  implies  $\deg P < k$ .

We conclude that  $p_k^\# = p_k + \dots$

**Proposition 3.5.** *Let  $k = 1, 2, \dots$ . In the weight grading, the top homogeneous component of  $p_k^\#$  has weight  $k + 1$  and can be written as*

$$-\frac{1}{k} [t^{k+1}] \left\{ \exp \left( -k \sum_{j=2}^{\infty} \frac{\tilde{p}_j}{j} t^j \right) \right\} \quad (34)$$

$$= \frac{\tilde{p}_{k+1}}{k+1} + \left\langle \text{a homogeneous polynomial in } \tilde{p}_2, \dots, \tilde{p}_k \right\rangle \text{ of total weight } k+1. \quad (35)$$

*Proof.* Apply (33) and recall that  $\text{wt}(p_j) = j + 1$ , because

$$p_j = \frac{\tilde{p}_{j+1}}{j+1} + \langle \text{a linear combination of } \tilde{p}_2, \dots, \tilde{p}_j \rangle.$$

As in the situation of Proposition 3.4, we may neglect the epsilon factors, which affect only terms of lower weight. For the same reason, we may replace each  $p_j$  by  $\tilde{p}_{j+1}/(j+1)$ . This leads to (34), and (35) follows from (34).

In Proposition 3.7 we invert the result of Proposition 3.5. Beforehand we state the following general fact.

**Proposition 3.6.** *Let  $a_2, a_3, \dots$  and  $b_2, b_3, \dots$  be two families of elements in a commutative algebra. Let*

$$A(t) = 1 + \sum_{j=2}^{\infty} a_j t^j, \quad B(u) = 1 + \sum_{j=2}^{\infty} b_j u^j$$

*be their generating series, and set*

$$\tilde{A}(t) = \ln A(t) = \sum_{j=2}^{\infty} \frac{\tilde{a}_j}{j} t^j.$$

*Then the following conditions are equivalent:*

(i) *The formal transformations  $x \rightarrow xA(x)$  and  $x \rightarrow x/B(x)$  are inverse to each other.*

$$(ii) \quad b_k = -\frac{1}{k-1} [t^k] \{A^{-(k-1)}(t)\}, \quad k = 2, 3, \dots$$

$$(iii) \quad a_k = \frac{1}{k+1} [u^k] \{B^{k+1}(u)\}, \quad k = 2, 3, \dots$$

$$(iv) \quad \tilde{a}_k = [u^k] \{B^k(u)\}, \quad k = 2, 3, \dots$$

*Proof.* This is a variation of Lagrange's inversion formula and can be proved by the standard argument, see, e.g., [44], [28], Example I.2.24.

In Proposition 3.7 we use only a part of the claims of Proposition 3.6. Another part will be used later on.

**Proposition 3.7.** *For  $k = 2, 3, \dots$*

$$\begin{aligned} \tilde{p}_k &= [u^k] \left\{ \left( 1 + \sum_{j=2}^{\infty} p_{j-1}^{\#} u^j \right)^k \right\} + \dots \\ &= \sum_{\substack{m_2, m_3, \dots \\ 2m_2 + 3m_3 + \dots = k}} \frac{k \downarrow \sum m_i}{\prod m_i!} \prod_{i \geq 2} (p_{i-1}^{\#})^{m_i} + \dots, \end{aligned} \quad (36)$$

*where dots mean a polynomial in  $p_1^{\#}, p_2^{\#}, \dots, p_{k-2}^{\#}$  of total weight  $\leq k-1$ , where  $\text{wt}(p_i^{\#}) = i+1$ .*

*Proof.* Assume that  $\tilde{a}_2, \tilde{a}_3, \dots$  and  $b_2, b_3, \dots$  are elements of a commutative algebra such that

$$b_k = \frac{1}{k} \tilde{a}_k + X_k(\tilde{a}_2, \dots, \tilde{a}_{k-1}), \quad k = 2, 3, \dots, \quad (37)$$

where  $X_k$  is an inhomogeneous polynomial in  $k - 2$  variables such that  $\text{wt}(X_k) \leq k$ , where  $\text{wt}(X_k)$  denotes the total weight counted with the convention that  $\text{wt}(\tilde{a}_j) = j$ .

Then, as is readily seen,

$$\frac{1}{k} \tilde{a}_k = b_k + Y_k(b_2, \dots, b_{k-1}), \quad k = 2, 3, \dots, \quad (38)$$

where, likewise,  $Y_k$  is a polynomial of total weight  $\leq k$ , with the convention that  $\text{wt}(b_j) = j$ .

Moreover, the top weight homogeneous component of  $Y_k$  depends only on the top weight homogeneous components of  $X_2, \dots, X_k$ .

Now let us set

$$\begin{aligned} \tilde{a}_k &= \tilde{p}_k, \quad b_k = p_{k-1}^\#, \quad k = 2, 3, \dots, \\ \tilde{A}(t) &= \sum_{j=2}^{\infty} \frac{\tilde{a}_j}{j} t^j, \quad A(t) = \exp \tilde{A}(t), \quad B(u) = 1 + \sum_{j=2}^{\infty} b_j u^j. \end{aligned}$$

By Proposition 3.5, we have

$$b_k = -\frac{1}{k-1} [t^k] \{A^{-(k-1)}(t)\} + \dots, \quad k = 2, 3, \dots,$$

where dots mean terms of lower weight. By (36), these relations are of the form (37). Therefore, to evaluate the inverse relations (38) up to lower weight terms, we may use formula (iv) of Proposition 3.6. This yields

$$\tilde{a}_k = [u^k] \{B^k(u)\} + \dots, \quad k = 2, 3, \dots,$$

which is exactly (36).

#### 4. The basis $\{p_\rho^\#\}$ and filtrations in $\mathbb{A}$

In this section we review some results of [25] and [15].

**Definition 4.1.** *To any partition  $\rho$  we assign a function  $p_\rho^\#$  on  $\mathbb{Y}$  as follows. Let  $r = |\rho|$ , let  $\lambda \in \mathbb{Y}$ , and denote  $n = |\lambda|$ . Then*

$$p_\rho^\#(\lambda) = \begin{cases} n^{\downarrow r} \cdot \frac{\chi_{\rho \cup 1^{n-r}}^\lambda}{\dim \lambda}, & n \geq r, \\ 0, & n < r, \end{cases}$$

where  $\rho \cup 1^{n-r} = (\rho, 1, \dots, 1) \in \mathbb{Y}_n$ .

When  $\rho$  consists of a single part,  $\rho = (r)$ , then this reduces to Definition 3.1.

Given a partition  $\rho$ , we shall denote by  $m_i = m_i(\rho)$  the multiplicity of  $i$  in  $\rho$ :

$$m_i(\rho) = \text{Card}\{j \mid \rho_j = i\}, \quad i = 1, 2, \dots .$$

By  $\ell(\rho)$  we denote the number of nonzero parts of  $\rho$  (the length of  $\rho$ ). We have  $\ell(\rho) = \sum m_i(\rho)$ .

Similarly to the conventional notation for the algebra  $\Lambda$ , we set

$$p_\rho = p_{\rho_1} \cdots p_{\rho_{\ell(\rho)}} = \prod_i p_i^{m_i(\rho)} .$$

The elements  $p_\rho$  form a homogeneous (in the canonical grading) basis in the algebra  $\mathbb{A}$ . Note that  $\deg p_\rho = |\rho|$ .

The next result generalizes Proposition 3.3 (first claim) and Proposition 3.4. It was first announced in [39].

**Proposition 4.2.** *For any partition  $\rho$ , the function  $p_\rho^\#$  introduced in Definition 4.1 is an element of  $\mathbb{A}$ . In the canonical grading, the top degree homogeneous component of  $p_\rho^\#$  equals  $p_\rho$ .*

*Proof.* Different proofs are given in [25] and [30]. See also [15], [26], [31].

**Corollary 4.3.** *The elements  $p_\rho^\#$  form a basis in  $\mathbb{A}$ .*

Note that this basis is inhomogeneous both in the canonical grading and the weight grading.

Given two partitions  $\sigma, \tau$ , we denote by  $\sigma \cup \tau$  the partition obtained by joining the parts of both partitions and then arranging them in descending order. In other words,  $\sigma \cup \tau$  is characterized by

$$m_i(\sigma \cup \tau) = m_i(\sigma) + m_i(\tau), \quad i = 1, 2, \dots .$$

**Corollary 4.4.** *For any partitions  $\sigma, \tau$ ,*

$$p_\sigma^\# p_\tau^\# = p_{\sigma \cup \tau}^\# + \dots ,$$

*where dots mean lower degree terms with respect to the canonical grading.*

Here and in what follows we define the degree of an inhomogeneous element as the maximal degree of its nonzero homogeneous components. In other words, we switch from the grading to the corresponding filtration.

Later on it will be shown that the claim of Corollary 4.4 also holds for the weight grading (or filtration), see Propositions 4.9 and 4.10.

Let  $f_{\sigma\tau}^\rho$  denote the structure constants of the algebra  $\mathbb{A}$  in the basis  $\{p_\rho^\#\}$ . I.e.,

$$p_\sigma^\# p_\tau^\# = \sum_{\rho} f_{\sigma\tau}^\rho p_\rho^\#.$$

By Corollary 4.4,  $f_{\sigma\tau}^\rho \neq 0$  implies  $|\rho| \leq |\sigma| + |\tau|$ . Moreover,

$$f_{\sigma\tau}^{\sigma\cup\tau} = 1. \quad (39)$$

Recall the conventional notation [28], §I.2:

$$z_\rho = \prod_i i^{m_i(\rho)} m_i(\rho)!$$

**Proposition 4.5.** *Let  $\rho, \sigma, \tau$  be arbitrary partitions. We have*

$$f_{\sigma\tau}^\rho = \frac{z_\sigma z_\tau}{z_\rho} g_{\sigma\tau}^\rho,$$

where  $g_{\sigma\tau}^\rho$  can be evaluated as follows.

Fix a set  $X$  of cardinality  $|\rho|$  and a permutation  $s : X \rightarrow X$  whose cycle structure is given by  $\rho$ . Then  $g_{\sigma\tau}^\rho$  equals the number of quadruples  $(X_1, s_1, X_2, s_2)$  such that:

- (i)  $X_1 \subseteq X, \quad X_2 \subseteq X, \quad X_1 \cup X_2 = X.$
- (ii)  $|X_1| = |\sigma|$  and  $s_1 : X_1 \rightarrow X_1$  is a permutation of cycle structure  $\sigma$ .
- (iii) Likewise,  $|X_2| = |\tau|$  and  $s_2 : X_2 \rightarrow X_2$  is a permutation of cycle structure  $\tau$ .
- (iv) Denote by  $\bar{s}_1 : X \rightarrow X$  and  $\bar{s}_2 : X \rightarrow X$  the natural extensions of  $s_{1,2}$  from  $X_{1,2}$  to the whole  $X$ . I.e.,  $\bar{s}_{1,2}$  is trivial on  $X \setminus X_{1,2}$ . Then the condition is that  $\bar{s}_1 \bar{s}_2 = s$ .

*Proof.* See [15], Proposition 6.2 and Theorem 9.1.

**Definition 4.6.** Fix an arbitrary subset  $J \subseteq \mathbb{N}$ , where  $\mathbb{N} = \{1, 2, \dots\}$ . For any partition  $\rho$ , set

$$|\rho|_J = |\rho| + \sum_{j \in J} m_j(\rho).$$

In particular, in the two extreme cases we have

$$|\rho|_\emptyset = |\rho|, \quad |\rho|_{\mathbb{N}} = |\rho| + \ell(\rho).$$

Next, following [15], define a filtration of the vector space  $\mathbb{A}$  by setting

$$\deg_J(p_\rho^\#) = |\rho|_J$$

and, more generally, for any  $f = \sum_{\rho} f_{\rho} p_{\rho}^{\#} \in \mathbb{A}$ ,

$$\deg_J(f) = \max_{\rho: f_{\rho} \neq 0} |\rho|_J.$$

**Proposition 4.7.** *For any  $J \subseteq \mathbb{N}$ , the filtration by  $\deg_J(\cdot)$  as defined above is compatible with the multiplication in  $\mathbb{A}$ . I.e., for any partitions  $\rho, \sigma, \tau$ ,*

$$f_{\sigma\tau}^{\rho} \neq 0 \implies |\rho|_J \leq |\sigma|_J + |\tau|_J,$$

so that this is an algebra filtration.

*Proof.* The argument presented below is a slightly rewritten version of that given in [15], Proposition 10.3.

Assume we are given partitions  $\rho, \sigma, \tau$  such that  $f_{\sigma\tau}^{\rho} \neq 0$ . Fix a set  $X$  and a permutation  $s : X \rightarrow X$  as in the statement of Proposition 4.5. By that proposition, there exists a quadruple  $\{X_1, s_1, X_2, s_2\}$  satisfying the four conditions (i)–(iv). Fix any such quadruple.

Decompose each of the permutations  $s, s_1, s_2$  into cycles and denote by  $C_J(\cdot)$  the set of all cycles whose lengths belong to the set  $J$ . Write

$$C_J(s_1) = A_J(s_1) \sqcup B_J(s_1), \quad C_J(s_2) = A_J(s_2) \sqcup B_J(s_2), \quad (40)$$

where  $A_J(s_1) \subseteq C_J(s_1)$  denotes the subset of those cycles of  $s_1$  that are entirely contained in  $X_1 \setminus X_2$ , while  $B_J(s_1) \subseteq C_J(s_1)$  denotes the subset of those cycles of  $s_1$  that have a nonempty intersection with  $X_1 \cap X_2$ . (Note that we count fixed points viewed as cycles of length 1, provided that  $1 \in J$ .) The sets  $A_J(s_2)$  and  $B_J(s_2)$  are defined similarly.

In this notation we have

$$C_J(s) = A_J(s_1) \sqcup A_J(s_2) \sqcup B_J(s), \quad (41)$$

where  $B_J(s)$  is the set of those cycles in  $C_J(s)$  that intersect both  $X_1$  and  $X_2$ .

Remark that

$$|\rho|_J = |X| + |C_J(s)|, \quad |\sigma|_J = |X_1| + |C_J(s_1)|, \quad |\tau|_J = |X_2| + |C_J(s_2)|.$$

Therefore, the required inequality  $|\rho|_J \leq |\sigma|_J + |\tau|_J$  means

$$|X| + |C_J(s)| \leq |X_1| + |C_J(s_1)| + |X_2| + |C_J(s_2)|.$$

By virtue of (40) and (41) this is equivalent to

$$|X| + |B_J(s)| \leq |X_1| + |B_J(s_1)| + |X_2| + |B_J(s_2)|. \quad (42)$$

We shall establish a stronger inequality,

$$|X| + |B_J(s)| \leq |X_1| + |X_2|, \quad (43)$$

which is equivalent to

$$|B_J(s)| \leq |X_1 \cap X_2|. \quad (44)$$

To prove the latter inequality, we shall show that each cycle  $c \in B_J(s)$  contains a point of  $X_1 \cap X_2$ .

By the definition of  $B_J(s)$ ,  $c$  contains both points of  $X_1$  and of  $X_2$ . Therefore, there exist points  $x_1 \in X_1 \cap c$  and  $x_2 \in X_2 \cap c$  such that  $sx_1 = x_2$ . We claim that either  $x_1$  or  $x_2$  lies in  $X_1 \cap X_2$ . Indeed, if  $x_1 \in X_1 \setminus X_2$  then

$$x_2 = sx_1 = \bar{s}_1 \bar{s}_2 x_1 = \bar{s}_1 x_1 = s_1 x_1 \in X_1.$$

This shows that  $x_2 \in X_1 \cap X_2$ , which completes the proof.

**Corollary 4.8.** (of the proof) *Let  $f_{\sigma\tau}^\rho \neq 0$  and  $|\rho|_J = |\sigma|_J + |\tau|_J$ . Then, in the notation of the proof of Proposition 4.7,  $B_J(s_1) = \emptyset$ ,  $B_J(s_2) = \emptyset$ , and (44) is actually an equality.*

*Proof.* Indeed, the equality  $|\rho|_J = |\sigma|_J + |\tau|_J$  means that (42) is an equality. Then (43) is an equality, too. This implies all the claims.

**Proposition 4.9.** *Assume  $J = \mathbb{N}$ . For any partitions  $\sigma, \tau$ ,*

$$p_\sigma^\# p_\tau^\# = p_{\sigma \cup \tau}^\# + \langle \text{a linear combination of } p_\rho^\# \text{'s with } |\rho|_{\mathbb{N}} < |\sigma|_{\mathbb{N}} + |\tau|_{\mathbb{N}} \rangle.$$

*Proof.* We have

$$p_\sigma^\# p_\tau^\# = \sum_{\rho} f_{\sigma\tau}^\rho p_\rho^\#.$$

By Proposition 4.7, only partitions  $\rho$  with  $|\rho|_{\mathbb{N}} \leq |\sigma|_{\mathbb{N}} + |\tau|_{\mathbb{N}}$  can really contribute.

By Corollary 4.8, if  $f_{\sigma\tau}^\rho \neq 0$  and  $|\rho|_{\mathbb{N}} = |\sigma|_{\mathbb{N}} + |\tau|_{\mathbb{N}}$ , then both  $B_{\mathbb{N}}(s_1)$  and  $B_{\mathbb{N}}(s_2)$  are empty, which implies  $X_1 \cap X_2 = \emptyset$ . Therefore,  $\rho = \sigma \cup \tau$ . Finally, by formula (39), which we have derived from Corollary 4.4,  $f_{\sigma\tau}^{\sigma \cup \tau} = 1$ . This completes the proof.

Note that formula (39) can also be obtained from Proposition 4.9.

**Proposition 4.10.** *The filtration of  $\mathbb{A}$  defined by  $\deg_{\mathbb{N}}(\cdot)$  coincides with the weight filtration.*

*Proof.* For any  $r = 0, 1, \dots$ , let  $\mathbb{A}'_r \subset \mathbb{A}$  denote the  $r$ th member of the first filtration, and let  $\mathbb{A}''_r \subset \mathbb{A}$  has the same meaning for the second filtration. Recall that

$$\begin{aligned}\mathbb{A}'_r &= \text{span}\{p_\rho^\# \mid |\rho|_{\mathbb{N}} = |\rho| + \ell(\rho) \leq r\}, \\ \mathbb{A}''_r &= \text{span}\{\tilde{p}_{k_1} \dots \tilde{p}_{k_l} \mid k_1, \dots, k_l \geq 2, \quad k_1 + \dots + k_l \leq r\}.\end{aligned}$$

Clearly,

$$\mathbb{A}'_0 = \mathbb{A}''_0 = \mathbb{A}'_1 = \mathbb{A}''_1 = \mathbb{R} \cdot 1.$$

We shall prove that for any  $r \geq 2$ , both  $\mathbb{A}'_r \subseteq \mathbb{A}''_r$  and  $\mathbb{A}''_r \subseteq \mathbb{A}'_r$ . By Proposition 4.9,

$$\mathbb{A}'_r = \text{span}\{p_{k_1}^\# \dots p_{k_l}^\# \mid k_1, \dots, k_l \geq 1, \quad k_1 + \dots + k_l + l \leq r\}.$$

Therefore, it suffices to show that

$$\begin{aligned}p_k^\# &\in \text{span}\{\tilde{p}_{k_1} \dots \tilde{p}_{k_l} \mid k_1, \dots, k_l \geq 2, \quad k_1 + \dots + k_l \leq k + 1\}, \\ \tilde{p}_k &\in \text{span}\{p_{k_1}^\# \dots p_{k_l}^\# \mid k_1, \dots, k_l \geq 1, \quad k_1 + \dots + k_l + l \leq k\}.\end{aligned}$$

The first inclusion follows from Proposition 3.5, and the second inclusion follows from Proposition 3.7.

In the remaining part of the section we focus on the filtration corresponding to  $J = \{1\}$ . It first appeared in [21], and we propose to call it the *Kerov filtration* of  $\mathbb{A}$ .

Let us abbreviate

$$|\rho|_1 = |\rho|_{\{1\}} = |\rho| + m_1(\rho), \quad \text{deg}_1(\cdot) = \text{deg}_{\{1\}}(\cdot).$$

The next three results will be used in Section 6.

**Proposition 4.11.** *For any partition  $\sigma$ ,*

$$p_\sigma^\# p_1^\# = p_{\sigma \cup 1}^\# + \langle \text{a term of lower degree with respect to } \text{deg}_1(\cdot) \rangle.$$

*Proof.* Actually, the following exact formula holds:

$$p_\sigma^\# p_1^\# = p_{\sigma \cup 1}^\# + |\sigma| \cdot p_\sigma^\#. \quad (45)$$

To prove this, apply Definition 4.1 and evaluate both sides at a partition  $\lambda$ . It suffices to assume that  $n = |\lambda|$  is large enough,  $n \geq |\sigma| + 1$ . Then we get, abbreviating  $k = |\sigma|$ ,

$$p_\sigma^\#(\lambda) = n^{\downarrow k} \cdot \frac{\chi_{\sigma \cup 1^{n-k}}^\lambda}{\dim \lambda}, \quad p_{\sigma \cup 1}^\#(\lambda) = n^{\downarrow k+1} \cdot \frac{\chi_{\sigma \cup 1^{n-k}}^\lambda}{\dim \lambda},$$

because  $(\sigma \cup 1) \cup 1^{n-k-1} = \sigma \cup 1^{n-k}$ .

Therefore, the verification of (45) at  $\lambda$  reduces to the relation

$$n^{\downarrow k} \cdot n = n^{\downarrow(k+1)} + n^{\downarrow k} \cdot k,$$

which is trivial.

Note that Proposition 4.11 can also be obtained from Proposition 4.7. We shall use this approach in the next proposition.

**Proposition 4.12.** *For any partition  $\sigma$  and any  $k \geq 2$ ,*

$$p_{\sigma}^{\#} p_k^{\#} = p_{\sigma \cup k}^{\#} + \begin{cases} k \cdot m_k(\sigma) \cdot p_{(\sigma \setminus k) \cup 1^k}^{\#} + \dots, & m_k(\sigma) \geq 1, \\ \dots, & m_k(\sigma) = 0, \end{cases} \quad (46)$$

where the partition  $\sigma \setminus k$  is obtained from  $\sigma$  by removing one part equal to  $k$ , i.e.,

$$m_i(\sigma \setminus k) = \begin{cases} m_i(\sigma), & i \neq k, \\ m_k(\sigma) - 1, & i = k, \end{cases}$$

and dots mean terms of lower degree, i.e., with  $\deg_1(\cdot) < |\sigma|_1 + k$ .

*Proof.* Assume first that  $\tau$  is an arbitrary partition (not necessarily  $\tau = (k)$ ) and search for partitions  $\rho$  such that  $f_{\sigma\tau}^{\rho} \neq 0$  and  $|\rho|_1 = |\sigma|_1 + |\tau|_1$ . Let us employ the notation introduced in the proof of Proposition 4.7 and apply Corollary 4.8. We get  $B_{\{1\}}(s_1) = \emptyset$ ,  $B_{\{1\}}(s_2) = \emptyset$ , and  $|B_{\{1\}}(s)| = |X_1 \cap X_2|$ . This means that in  $X_1 \cap X_2$  there is no 1-cycle (=fixed point) for  $s_1$  and  $s_2$ , but all points of  $X_1 \cap X_2$  are fixed by  $s$ .

This shows that either  $X_1 \cap X_2 = \emptyset$  or  $X_1 \cap X_2$  entirely consists of common nontrivial cycles of the permutations  $s_1$  and  $s_2^{-1}$ .

Now apply this conclusion to the special case  $\tau = (k)$  that we need. Recall that  $k \geq 2$ . The first possibility,  $X_1 \cap X_2 = \emptyset$ , means that  $\rho = \sigma \cup \tau = \sigma \cup (k)$ . Then the corresponding coefficient  $f_{\sigma\tau}^{\rho}$  equals 1, see (39). This explains the term  $p_{\sigma \cup k}^{\#}$  in the right-hand side of (46).

The second possibility means that  $X_1 \supseteq X_2$ , because  $s_2^{-1}$  reduces to a single  $k$ -cycle, which is also a  $k$ -cycle of  $s_1$ . This implies that  $m_k(\sigma) \geq 1$  and  $\rho = (\sigma \setminus k) \cup 1^k$ .

It remains to evaluate the coefficient  $f_{\sigma\tau}^{\rho}$ . Let us abbreviate  $m = m_k(\sigma)$ ,  $l = m_1(\sigma)$ . We must prove that  $f_{\sigma\tau}^{\rho} = km$ . To do this we apply Proposition 4.5. We readily get

$$\frac{z_{\rho}}{z_{\sigma} z_{\tau}} = \frac{(k+l)!}{l! k^2 m},$$

so that  $f_{\sigma\tau}^\rho = km$  is equivalent to

$$g_{\sigma\tau}^\rho = \frac{(k+l)!}{l!k}.$$

Let us check this. By the definition of  $g_{\sigma\tau}^\rho$ , in our situation it equals the number of ways to choose a  $k$ -cycle inside a  $(k+l)$ -point set. This number equals

$$\frac{(k+l)!}{k!l!} \cdot (k-1)! = \frac{(k+l)!}{l!k},$$

(the number of  $k$ -point subsets inside a  $(k+l)$ -point set, times the number of different  $k$ -cycle structures on a given  $k$ -point set). This concludes the proof.

Finally, note that

$$\text{inv}(p_\rho^\#) = (-1)^{|\rho|+\ell(\rho)} p_\rho^\#. \quad (47)$$

Indeed, this follows from the definition of  $p_\rho^\#$ . In particular,

$$\text{inv}(p_k^\#) = (-1)^{k-1} p_k^\#. \quad (48)$$

This symmetry property will be used in the proofs of Proposition 7.3 and Theorem 10.2.

**Corollary 4.13.** (of the proof) *Let  $\sigma$  and  $\tau$  be two partitions with no common part, i.e., for any  $i = 1, 2, \dots$ , at least one of the multiplicities  $m_i(\sigma)$ ,  $m_i(\tau)$  vanishes. Then*

$$p_\sigma^\# p_\tau^\# = p_{\sigma \cup \tau}^\# + \langle \text{terms of lower degree } \deg_1(\cdot) \rangle.$$

*Proof.* Let  $\rho$ ,  $X_1$ ,  $X_2$  be as in the beginning of the proof of Proposition 4.12. Recall the claim stated in the second paragraph of that proof: either  $X_1 \cap X_2 = \emptyset$  or  $X_1 \cap X_2$  entirely consists of common nontrivial cycles of the permutations  $s_1$  and  $s_2^{-1}$ . The second possibility contradicts the assumption that  $\sigma$  and  $\tau$  have no common part. Hence the first possibility holds, which means that  $\rho = \sigma \cup \tau$ . We know that the corresponding coefficient  $f_{\sigma\tau}^\rho$  equals 1, which concludes the proof.

## 5. The Plancherel measure and the law of large numbers

Consider the set  $\mathbb{Y}_n$  of Young diagrams with  $n$  boxes,  $n = 1, 2, \dots$ , and equip it with the measure  $M_n$ , defined by

$$M_n(\lambda) = \frac{\dim^2 \lambda}{n!}, \quad \lambda \in \mathbb{Y}_n.$$

This is a probability measure, because, by Burnside's theorem,

$$\sum_{\lambda \in \mathbb{Y}_n} \dim^2 \lambda = |\mathfrak{S}_n| = n!.$$

It is called the *Plancherel measure*, see [38], [39], [40] for more details.

Given a function  $f$  on  $\mathbb{Y}_n$ , we define by  $\langle f \rangle_n$  its expectation with respect to  $M_n$ . That is,

$$\langle f \rangle_n = \sum_{\lambda \in \mathbb{Y}_n} f(\lambda) M_n(\lambda) = \frac{1}{n!} \sum_{\lambda \in \mathbb{Y}_n} f(\lambda) \dim^2 \lambda.$$

If  $f$  is a function on the whole set  $\mathbb{Y}$ , we write  $\langle f \rangle_n$  instead of  $\langle f |_{\mathbb{Y}_n} \rangle_n$ . We shall use this convention for functions  $f \in \mathbb{A}$ . In this way we get the family of linear functionals  $\langle \cdot \rangle_n$ ,  $n = 1, 2, \dots$ , on the algebra  $\mathbb{A}$ . These functionals have a very simple form on the basis  $\{p_\rho^\#\}$ .

**Proposition 5.1.** *For any partition  $\rho$ ,*

$$\langle p_\rho^\# \rangle_n = \begin{cases} n^{\downarrow r}, & \rho = (1^r), \quad r = 1, 2, \dots, \\ 0, & \rho \neq (1^r). \end{cases}$$

*Proof.* Set  $r = |\rho|$ . If  $n < r$  then  $p_\rho^\#$  vanishes on  $\mathbb{Y}_n$ , which agrees with the formula in question, because  $n^{\downarrow r} = 0$  whenever  $n < r$ .

Assume  $n \geq r$ . By the definition of  $p_\rho^\#$ , see Definition 4.1,

$$\langle p_\rho^\# \rangle_n = n^{\downarrow r} \cdot \frac{1}{n!} \cdot \sum_{\lambda \in \mathbb{Y}_n} \chi_{\rho \cup 1^{n-r}}^\lambda \dim \lambda.$$

Remark that the sum above equals the value of the regular character (i.e., the character of the regular representation of  $\mathfrak{S}_n$ ) on the conjugacy class  $\rho \cup 1^{n-r}$ . But the regular character is the delta function at  $\{e\} \subset \mathfrak{S}_n$ , multiplied by  $n!$ . Therefore, the sum in question vanishes unless  $\rho \cup 1^{n-r}$  is the trivial class (i.e.,  $\rho$  itself is trivial,  $\rho = (1^r)$ ), in which case the sum equals  $n!$ . Consequently, we get  $n^{\downarrow r}$ .

**Proposition 5.2.** *For any  $f \in \mathbb{A}$ , the expectation  $\langle f \rangle_n$  is a polynomial in  $n$ . The degree of this polynomial is bounded from above by  $\frac{1}{2} \deg_1(f)$ .*

*Proof.* It suffices to check this for  $f = p_\rho^\#$ . Let  $r = |\rho|$ . If  $\rho \neq (1^r)$  then  $\langle f \rangle_n \equiv 0$ , by virtue of Proposition 5.1. If  $\rho = (1^r)$  then, by Proposition 5.1,  $\langle f \rangle_n = n^{\downarrow r}$ . On the other hand,  $\deg_1(f) = 2r$ , which agrees with the claim.

We define the function  $\Omega(x)$  on  $\mathbb{R}$  by

$$\Omega(x) = \begin{cases} \frac{2}{\pi} (x \arcsin \frac{x}{2} + \sqrt{4-x^2}), & |x| \leq 2, \\ |x|, & |x| \geq 2. \end{cases}$$

Note that both expressions agree at  $x = \pm 2$ , so that  $\Omega(x)$  is continuous. Moreover, the first derivative  $\Omega'(x)$  is continuous on the whole  $\mathbb{R}$ , while  $\Omega''(x)$  is not. This is clear from the explicit expressions

$$\Omega'(x) = \frac{2}{\pi} \arcsin \frac{x}{2}, \quad \Omega''(x) = \frac{2}{\pi} \frac{1}{\sqrt{4-x^2}}, \quad |x| < 2.$$

Next, we have  $|\Omega'(x)| < 1$  for  $|x| < 2$ , which implies that  $\Omega(x)$  belongs to  $\mathcal{D}^0$ . Applying (13) for  $\omega = \Omega$  we get

$$\tilde{p}_k[\Omega] = -k \int_{\mathbb{R}} x^{k-1} \left( \frac{\Omega(x) - |x|}{2} \right)' dx = -\frac{k}{2} \int_{-2}^2 x^{k-1} \left( \frac{2}{\pi} \arcsin \frac{x}{2} - \operatorname{sgn} x \right) dx.$$

**Proposition 5.3.** *We have*

$$\tilde{p}_k[\Omega] = \begin{cases} \frac{(2m)!}{m!m!}, & k = 2m, \quad m = 1, 2, \dots, \\ 0, & k = 1, 3, 5, \dots \end{cases}$$

*Proof.* Since  $\Omega(x)$  is even,  $\tilde{p}_k[\Omega] = 0$  for odd  $k$ . For even  $k$  we get from (13)

$$\tilde{p}_{2m}[\Omega] = \int_0^2 \left( 1 - \frac{2}{\pi} \arcsin \frac{x}{2} \right) d(x^{2m}).$$

Setting  $x = 2 \sin \theta$  and integrating by parts we get the result.

We proceed to the “law of large numbers” for the Plancherel measures  $M_n$ . Actually, it is implied by the “central limit theorem” which will be established in Section 7. However, we prefer to give here an independent short proof.

Recall that in Definition 2.3 we have attached to any Young diagram  $\lambda \in \mathbb{Y}_n$  a function  $\lambda(\cdot) \in \mathcal{D}^0$ . We define now a scaled version of it:

$$\bar{\lambda}(x) = n^{-1/2} \lambda(n^{1/2}x), \quad x \in \mathbb{R}, \quad \lambda \in \mathbb{Y}_n.$$

This is a special case of Definition 2.10. The correspondence  $\lambda \mapsto \bar{\lambda}(\cdot)$  provides an embedding  $\mathbb{Y}_n \hookrightarrow \mathcal{D}^0$ .

**Theorem 5.4.** (Law of large numbers, 1st form) *Let  $\lambda$  range over  $\mathbb{Y}_n$ , and let us view  $\bar{\lambda}(\cdot)$  as a random function defined on the probability space  $(\mathbb{Y}_n, M_n)$ , where  $M_n$  is the Plancherel measure. Let  $\Omega$  be as above. Then we have*

$$\lim_{n \rightarrow \infty} \int (\bar{\lambda}(x) - \Omega(x)) x^k dx = 0 \quad \text{in probability, for any } k = 0, 1, \dots \quad (49)$$

*Proof.* Let  $\bar{M}_n$  be the pushforward of  $M_n$  under the embedding  $\mathbb{Y}_n \hookrightarrow \mathcal{D}^0$ . Then  $\bar{M}_n$  is a probability measure on the space  $\mathcal{D}^0$ . Given a “test” function  $f$  on  $\mathcal{D}^0$ , let  $\langle f, \bar{M}_n \rangle$  denote the result of pairing between  $f$  and  $\bar{M}_n$ :

$$\langle f, \bar{M}_n \rangle = \sum_{\lambda \in \mathbb{Y}_n} f(\bar{\lambda}(\cdot)) M_n(\lambda).$$

Recall that the elements of  $\mathbb{A}$  can be interpreted as functions on  $\mathcal{D}^0$ , see Definition 2.10 b). Let us take them as “test” functions.

We claim that

$$\lim_{n \rightarrow \infty} \langle f, \bar{M}_n \rangle = f[\Omega], \quad f \in \mathbb{A}. \quad (50)$$

In other words, the measures  $\bar{M}_n$  on the space  $\mathcal{D}^0$  converge to the Dirac measure at  $\Omega \in \mathcal{D}^0$  in the weak topology defined by the function algebra  $\mathbb{A}$ .

Let us prove (50). Without loss of generality we may assume that  $f$  is a homogeneous element with respect to the weight grading in  $\mathbb{A}$ . Then, by virtue of Proposition 2.11,

$$\langle f, \bar{M}_n \rangle = n^{-\text{wt}(f)/2} \langle f \rangle_n. \quad (51)$$

By Proposition 5.2,  $\langle f \rangle_n$  is a polynomial in  $n$  of degree less or equal to  $\frac{1}{2} \deg_1(f)$ . Note that  $\deg_1(\cdot) \leq \deg_{\mathbb{N}}(\cdot)$  and  $\deg_{\mathbb{N}}(\cdot)$  coincides with  $\text{wt}(\cdot)$ , see Proposition 4.10. Thus, the degree of  $\langle f \rangle_n$  does not exceed  $\frac{1}{2} \text{wt}(f)$ , which implies that (51) has a limit as  $n \rightarrow \infty$ .

Expand  $f$  in the basis  $\{p_\rho^\#\}$ :

$$f = \sum_{\rho} f_\rho p_\rho^\#.$$

By virtue of Proposition 5.1,

$$\lim_{n \rightarrow \infty} n^{-\text{wt}(f)/2} \langle f \rangle_n = f_{(1^{\text{wt}(f)/2})}$$

with the understanding that the symbol  $f_{(1^{k/2})}$  means 0 whenever  $k$  is odd.

Next, Proposition 4.9 (together with Proposition 4.10) implies the multiplicativity property

$$(fg)_{(1^{\text{wt}(fg)/2})} = f_{(1^{\text{wt}(f)/2})}g_{(1^{\text{wt}(g)/2})},$$

where  $f, g$  are arbitrary weight homogeneous elements. Consequently, it suffices to examine the case  $f = \tilde{p}_k$ , i.e., to show that

$$(\tilde{p}_k)_{(1^{k/2})} = \tilde{p}_k[\Omega].$$

The right-hand side was found in Proposition 5.3, while the left-hand side can be evaluated using Proposition 3.7. The result is the same, which concludes the proof of (50).

Now let us show that (50) implies (49). Indeed, (49) is equivalent to

$$\lim_{n \rightarrow \infty} \int \frac{\bar{\lambda}(x) - |x|}{2} x^k dx = \int \frac{\Omega(x) - |x|}{2} x^k dx \quad \text{in probability, for } k = 0, 1, \dots$$

By Proposition 2.2, this is equivalent to

$$\lim_{n \rightarrow \infty} \tilde{p}_k[\bar{\lambda}(\cdot)] = \tilde{p}_k[\Omega] \quad \text{in probability, for } k = 2, 3, \dots$$

Applying Chebyshev's inequality we see that to prove this, it suffices to check that the first and the second moments of the random variable  $\tilde{p}_k[\bar{\lambda}(\cdot)]$  converge, as  $n \rightarrow \infty$ , to  $\tilde{p}_k[\Omega]$  and  $\tilde{p}_k^2[\Omega]$ , respectively. But this is a particular case of (50) corresponding to  $f = \tilde{p}_k$  and  $f = \tilde{p}_k^2$ , respectively.

**Theorem 5.5.** (Law of large numbers, 2nd form) *Let  $\lambda$  range over  $\mathbb{Y}_n$ , and let us view  $\bar{\lambda}(\cdot)$  as a random function defined on the probability space  $(\mathbb{Y}_n, M_n)$ , where  $M_n$  is the Plancherel measure. Let  $\Omega$  be as above. Then we have*

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |\bar{\lambda}(x) - \Omega(x)| = 0 \quad \text{in probability.} \quad (52)$$

We need two lemmas.

**Lemma 5.6.** *There exists an interval  $I \subset \mathbb{R}$  such that the probability that  $\bar{\lambda}(x) - |x|$  is supported by  $I$  tends to 1 as  $n \rightarrow \infty$ .*

*Proof.* This follows from a finer result due to Hammersley [13]. He has proved that there exists a constant  $c$  such that for any  $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} M_n\{\lambda \in \mathbb{Y}_n \mid |\lambda_1 - c\sqrt{n}| < \varepsilon, \quad |\lambda'_1 - c\sqrt{n}| < \varepsilon\} = 1.$$

Actually, the constant  $c$  equals 2 (this was first proved by Vershik–Kerov [38], [40]), and at present much more is known about the asymptotics of  $\lambda_1$ , see, e.g., the expository paper [1]. But, for our purpose, the old Hammersley's result is enough.

**Lemma 5.7.** Fix an interval  $I = [a, b] \subset \mathbb{R}$ , and let  $\Sigma$  denote the set of all real-valued functions  $\sigma(x)$  on  $\mathbb{R}$ , supported by  $I$  and satisfying the Lipschitz condition  $|\sigma(x_1) - \sigma(x_2)| \leq |x_1 - x_2|$ .

On the set  $\Sigma$ , the weak topology defined by the functionals

$$\sigma \mapsto \int \sigma(x)x^k dx, \quad k = 0, 1, \dots,$$

coincides with the uniform topology defined by the supremum norm  $\|\sigma\| = \sup |\sigma(x)|$ .

This fact was pointed out in [22], §2.5.

*Proof.* Clearly, the uniform topology is stronger than the weak topology. Let us check the inverse claim. Given  $x \in I$  and  $\varepsilon > 0$ , let

$$V(x, \varepsilon) = \{\sigma \in \Sigma \mid |\sigma(x)| \leq \varepsilon\}.$$

Pick points  $a = \varepsilon_1 < \dots < \varepsilon_n = b$  dividing  $I$  into subintervals of length  $\leq 2\varepsilon$ . Then, by the Lipschitz condition, the ball  $\|\sigma\| \leq 2\varepsilon$  contains the intersections of  $V(x_i, \varepsilon)$ 's. Hence the required claim reduces to the following one:

Fix  $x \in I$  and  $\varepsilon > 0$ . Then  $V(x, \varepsilon)$  contains a neighborhood of 0 in the weak topology.

Let us remark that functions  $\sigma \in \Sigma$  are uniformly bounded,  $\|\sigma\| \leq (b - a)/2$ . Hence the weak topology on  $\Sigma$  will not change if we take, as functionals, integrals with arbitrary continuous functions  $F$ . Now let us take a continuous function  $F(x) \geq 0$ , concentrated in the  $\varepsilon/2$ -neighborhood of  $x$  and such that  $\int F(y)dy = 1$ . We claim that

$$\left| \int \sigma(x)F(x)dx \right| \leq \varepsilon/2 \Rightarrow \sigma \in V(x, \varepsilon).$$

Indeed, assume that  $\sigma \notin V(x, \varepsilon)$ , i.e.,  $|\sigma(x)| > \varepsilon$ . Without loss of generality we may assume that  $\sigma(x) > \varepsilon$ . Then, for any  $y$  such that  $|x - y| \leq \varepsilon/2$ , we have  $\sigma(y) > \varepsilon/2$ , hence  $\int \sigma(y)F(y) > \varepsilon/2$ , which proves our claim.

*Proof of Theorem 5.5.* This immediately follows from Theorem 5.4 and Lemmas 5.6, 5.7.

## 6. The central limit theorem for characters

For any  $f \in \mathbb{A}$ , we denote by  $f^{(n)}$  the random variable defined on the probability space  $(\mathbb{Y}_n, M_n)$  and obtained by restricting  $f$  to  $\mathbb{Y}_n$ .

By the symbol  $\xrightarrow{d}$  we will denote convergence of random variables in distribution, see, e.g. [34].

The aim of this section is to prove the following result.

**Theorem 6.1.** (Central limit theorem for characters) *Choose a sequence  $\{\xi_k\}_{k=2,3,\dots}$  of independent standard Gaussian random variables. As  $n \rightarrow \infty$ , we have*

$$\left\{ \frac{p_k^{\#(n)}}{n^{k/2}} \right\}_{k=2,3,\dots} \xrightarrow{d} \{\sqrt{k}\xi_k\}_{k=2,3,\dots} \tag{53}$$

In more detail, for any fixed  $N = 2, 3, \dots$ , the joint distribution of  $N - 1$  random variables

$$\frac{p_k^{\#(n)}}{\sqrt{k} n^{k/2}}, \quad 2 \leq k \leq N,$$

weakly tends, as  $n \rightarrow \infty$ , to the standard Gaussian measure on  $\mathbb{R}^{N-1}$ . Note that we could take equally well in (53) the random variables

$$\frac{n^{k/2}}{\sqrt{k}} \frac{\chi_{(k, 1^{n-k})}^\lambda}{\dim \lambda}, \quad 2 \leq k \leq N,$$

where  $\lambda \in \mathbb{Y}_n$  is the random Plancherel diagram.

The proof of Theorem 6.1 will be given after some preparation work, based on Propositions 4.11, Proposition 4.12, and Corollary 4.13.

It will be convenient to extend the algebra  $\mathbb{A}$ : we adjoin to it the square root of the element  $p_1^\# = p_1$  and then localize over the multiplicative family generated by  $\sqrt{p_1^\#}$ . Let  $\mathbb{A}^{\text{ext}}$  denote the resulting algebra. As a basis in  $\mathbb{A}^{\text{ext}}$  one can take the elements of the form

$$p_\rho^\# \cdot (p_1^\#)^{m/2}, \quad m_1(\rho) = 0, \quad m \in \mathbb{Z}. \tag{54}$$

We equip  $\mathbb{A}^{\text{ext}}$  with a filtration by assigning to  $p_\rho^\# \cdot (p_1^\#)^{m/2}$  the degree  $\text{deg}_1(\cdot) = |\rho|_1 + m$ . That is, the  $N$ th term of the filtration is spanned by all basis elements (54) with  $|\rho|_1 + m \leq N$ . Here  $N$  ranges over  $\mathbb{Z}$ . On the subalgebra  $\mathbb{A} \subset \mathbb{A}^{\text{ext}}$ , this filtration agrees with that induced by the Kerov degree. Indeed, this claim follows from Proposition 4.11.

Since  $p_1^{\#(n)} \equiv n$ , the symbol  $f^{(n)}$  makes sense for any  $f \in \mathbb{A}^{\text{ext}}$ . Specifically, if  $f = g(p_1^\#)^{m/2}$  with  $g \in \mathbb{A}$  and  $m \in \mathbb{Z}$  then  $f^{(n)} = g^{(n)} \cdot n^{m/2}$ . Note also that Proposition 5.2 admits the following extension:

**Proposition 6.2.** *For any  $f \in \mathbb{A}^{\text{ext}}$ ,  $\langle f \rangle_n$  is a Laurent polynomial in  $n^{1/2}$  whose degree with respect to  $n$  is bounded from above by  $\frac{1}{2} \text{deg}_1(f)$ .*

*Proof.* Evident from Proposition 5.2 and the fact that  $p_1^{\#(n)} \equiv n$ .

Let  $H_m(x)$ , where  $m = 0, 1, 2, \dots$ , be the Hermite polynomials in the normalization of [37], [9]. We shall need slightly modified polynomials, which we denote by  $\mathcal{H}_m(x)$ :

$$\mathcal{H}_m(x) = 2^{-m/2} H_m(\sqrt{2}x) = m! \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{(-1/2)^j x^{m-2j}}{j!(m-2j)!}.$$

These are monic polynomials, which form the orthogonal system with respect to standard Gaussian measure  $(2\pi)^{-1/2} \exp(-x^2/2)dx$ . They are characterized by the recurrence relation

$$x\mathcal{H}_m = \mathcal{H}_{m+1} + m\mathcal{H}_{m-1} \quad (55)$$

together with the initial data  $\mathcal{H}_0 = 1$ ,  $\mathcal{H}_1 = x$ .

For an arbitrary partition  $\rho$ , we define the element  $\eta_\rho \in \mathbb{A}^{\text{ext}}$ , which is a normalization of  $p_\rho^\#$ :

$$\eta_\rho = \frac{p_\rho^\#}{(p_1^\#)^{m_1(\rho)} \prod_{k \geq 2} (k(p_1^\#)^k)^{m_k(\rho)/2}} = \frac{p_\rho^\#}{(p_1^\#)^{|\rho|/2} \prod_{k \geq 2} k^{m_k(\rho)/2}}. \quad (56)$$

Note that  $\deg_1(\eta_\rho) = 0$ .

We abbreviate  $\eta_k = \eta_{(k)}$ . Note that

$$\eta_k = \frac{p_k^\#}{\sqrt{k} (p_1^\#)^{k/2}}, \quad k = 2, 3, \dots \quad (57)$$

**Proposition 6.3.** *For any partition  $\rho$ , we have*

$$\eta_\rho = \prod_{k \geq 2} \mathcal{H}_{m_k(\rho)}(\eta_k) + \dots \quad (58)$$

where dots denote a remainder term with  $\deg_1(\cdot) < 0$ .

In particular,  $\eta_\rho$  does not depend, up to terms of negative degree, from the value of  $m_1(\rho)$ .

*Proof.* Examine first the particular case  $\rho = (k^m)$ , where  $k = 2, 3, \dots$  and  $m = 1, 2, \dots$ . Then our claim means that

$$\eta_{(k^m)} = \mathcal{H}_m(\eta_k) + \dots$$

Taking  $\sigma = (k^m)$  in Proposition 4.12 we get

$$p_{(k^m)}^\# \cdot p_k^\# = p_{(k^{m+1})}^\# + k m p_{(k^{m-1}, 1^k)}^\# + \dots = p_{(k^{m+1})}^\# + k m p_{(k^{m-1})}^\# \cdot (p_1^\#)^k + \dots$$

for any  $m \geq 1$ , where dots mean lower degree terms. This is equivalent to

$$\eta_{(k^m)} \cdot \eta_k = \eta_{(k^{m+1})} + m \eta_{(k^{m-1})} + \dots, \quad m \geq 1, \quad (59)$$

where dots mean terms of negative degree. Within these terms, (59) coincides with the recurrence relation (55), which proves (58).

The case of an arbitrary  $\rho$  is reduced to the particular case  $\rho = (k^m)$  using Corollary 4.13.

The next claim is a well-known general result. It justifies the moment method, which is a convenient tool for checking convergence in distribution.

**Proposition 6.4.** *Let  $a^{(n)}$  be a sequence of real random variables. Assume that  $a^{(n)}$  have finite moments of any order, and the moments converge, as  $n \rightarrow \infty$ , to the respective moments of a random variable  $a$ . Finally, assume that  $a$  is uniquely determined by its moments, which holds, e.g., if the characteristic function of  $a$  is analytic.*

*Then  $a^{(n)} \xrightarrow{d} a$ . Moreover, this claim also holds when the variables in question take vector values, i.e., when each  $a^{(n)}$ , as well as  $a$ , is a system of random variables.*

*Sketch of proof.* Let  $P^{(n)}$  denote the distribution of the random variable  $a^{(n)}$  and  $P$  be the distribution of  $a$ . We have to prove that  $P^{(n)}$  weakly converges to  $P$  as  $n \rightarrow \infty$ . The assumption on the moments implies that  $\{P^{(n)}\}$  is a tight family of probability measures on  $\mathbb{R}$ . So, it suffices to prove that any partial weak limit  $P'$  of the sequence  $\{P^{(n)}\}$  coincides with  $P$ . Using again the condition on the moments one can show that the moments of  $P'$  exist and coincide with the limits of the respective moments of  $\{P^{(n)}\}$ . Hence, these are exactly the moments of  $P$ . By the uniqueness assumption,  $P' = P$ .

For another proof, see Feller [11], Ch. VIII, §6, Example b.

*Proof of Theorem 6.1.* We must prove that

$$\{\eta_k^{(n)}\}_{k \geq 2} \xrightarrow{d} \{\xi_k\}_{k \geq 2}, \quad n \rightarrow \infty.$$

By Proposition 6.4, it suffices to check that

$$\left\langle \prod_{k \geq 2} \eta_k^{m_k} \right\rangle_n \xrightarrow{d} \left\langle \prod_{k \geq 2} \xi_k^{m_k} \right\rangle_{\text{Gauss}}, \quad n \rightarrow \infty, \quad (60)$$

for any finite collection  $\{m_k\}_{k \geq 2}$  of nonnegative integers, where the brackets  $\langle \cdot \rangle_{\text{Gauss}}$  mean expectation with respect to the standard Gaussian measure. The uniqueness hypothesis of Proposition 6.4 is clearly satisfied.

The limit relations (60) are equivalent to the following ones:

$$\left\langle \prod_{k \geq 2} \mathcal{H}_{m_k}(\eta_k) \right\rangle_n \xrightarrow{d} \prod_{k \geq 2} \langle \mathcal{H}_{m_k}(\xi_k) \rangle_{\text{Gauss}}$$

for any finite collection  $\{m_k \in \mathbb{Z}_+\}$ .

If all the numbers  $m_k$  are equal to 0 then the expressions in both sides equal 1, and there is nothing to prove. So, let us assume that some of the  $m_k$ 's are nontrivial. Then the right-hand side vanishes, because, for a standard Gaussian  $\xi$ ,

$$\langle \mathcal{H}_m(\xi) \rangle_{\text{Gauss}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{H}_m(x) e^{-x^2/2} dx = 0, \quad m = 1, 2, \dots,$$

by the orthogonal property of the polynomials  $\mathcal{H}_m$ .

Let us examine the left-hand side. Set  $\rho = (\prod_{k \geq 2} k^{m_k})$ . By Proposition 6.2,

$$\prod_{k \geq 2} \mathcal{H}_{m_k}(\eta_k) = \eta_\rho + \text{a "remainder term" of strictly negative degree.}$$

By our assumption,  $\rho$  is nonempty. Moreover,  $m_1(\rho) = 0$ , so that  $\rho \neq (1^r)$ . By Proposition 5.1,  $\langle \eta_\rho \rangle_n \equiv 0$ . Finally, by Proposition 6.2,

$$\langle \text{the "remainder term"} \rangle_n = O(n^{-1/2}). \quad (61)$$

This concludes the proof. □

Theorem 6.1 can be generalized as follows:

**Theorem 6.5.** *Let  $\rho$  range over the set of all partitions. We have*

$$\{\eta_\rho^{(n)}\} \xrightarrow{d} \left\{ \prod_{k \geq 2} \mathcal{H}_{m_k(\rho)}(\xi_k) \right\}, \quad n \rightarrow \infty,$$

where, as before,  $\xi_2, \xi_3, \dots$  are independent standard Gaussians.

*Proof.* The above argument shows that any mixed moment of the random variables from the left-hand side converges, as  $n \rightarrow \infty$ , to the respective moment of the random variables from the right-hand side. However, we cannot use the moment method, because a polynomial in Gaussian variables does not necessarily satisfy the uniqueness assumption mentioned in Proposition 6.4. For this reason we argue in a different way.

Assume that

$$\{a_1^{(n)}, a_2^{(n)}, \dots\} \xrightarrow{d} \{a_1, a_2, \dots\}, \quad n \rightarrow \infty,$$

where  $a^{(n)} = \{a_1^{(n)}, a_2^{(n)}, \dots\}$  are families of random variables depending on  $n$  and  $a = \{a_1, a_2, \dots\}$  is one more family of random variables. Next, assume that  $f_1(x) = f_1(x_1, x_2, \dots)$ ,  $f_2(x) = f_2(x_1, x_2, \dots)$ ,  $\dots$  are continuous functions in real variables  $x = (x_1, x_2, \dots)$ , where each function actually depends on finitely many variables only. Then

$$\{f_1(a^{(n)}), f_2(a^{(n)}), \dots\} \xrightarrow{d} \{f_1(a), f_2(a), \dots\}, \quad n \rightarrow \infty.$$

Using this general fact we conclude from Theorem 6.1 that any polynomial in  $\eta_2^{(n)}, \eta_3^{(n)}, \dots$  converges in distribution to the same polynomial in  $\xi_2, \xi_3, \dots$ . Moreover, this also holds for any finite system of polynomials. By virtue of (61), each  $\eta_\rho$  is a polynomial in  $\eta_2^{(n)}, \eta_3^{(n)}, \dots$ , within a “remainder term”. So we only need to check that the “remainder term” does not affect the convergence in distribution.

Remark that the “remainder term” is of the form  $r^{(n)}$ , where  $r$  is an element of  $\mathbb{A}^{\text{ext}}$  of strictly negative degree. It follows that any moment of  $r^{(n)}$  tends to 0 as  $n \rightarrow \infty$ , which implies that  $r^{(n)} \xrightarrow{d} 0$ . This shows that the “remainder term” is negligible.

For a different proof of Theorems 6.1 and 6.5, see [14].

## 7. The central limit theorem for Young diagrams

Given  $\lambda \in \mathbb{Y}_n$ , we set

$$\Delta_\lambda(x) = \frac{\sqrt{n}}{2}(\bar{\lambda}(x) - \Omega(x)), \quad x \in \mathbb{R}. \quad (62)$$

This is a continuous function on  $\mathbb{R}$  with compact support. Dropping  $\lambda$ , which we consider as the random element from the probability space  $(\mathbb{Y}_n, M_n)$ , we interpret (62) as a random function  $\Delta^{(n)}(x)$ .

For any polynomial  $v \in \mathbb{R}[x]$ , the integral

$$v^{(n)} = \int_{\mathbb{R}} v(x) \Delta^{(n)}(x) dx \quad (63)$$

makes sense (because  $\Delta^{(n)}$  is compactly supported) and is a random variable. We aim to show that the random variables (63), where  $v$  ranges over  $\mathbb{R}[x]$ , are asymptotically Gaussian.

The result will be stated in terms of the Chebyshev polynomials of the second kind. Instead of the conventional polynomials  $U_k(x)$  (see [37], [9]) we prefer to deal with slightly modified polynomials

$$u_k(x) = U_k(x/2) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} x^{k-2j}, \quad k = 0, 1, 2, \dots \quad (64)$$

Note that

$$u_k(2 \cos \theta) = \frac{\sin((k+1)\theta)}{\sin \theta} \quad (65)$$

and

$$\int_{-2}^2 u_k(x) u_l(x) \frac{\sqrt{4-x^2}}{2\pi} dx = \delta_{k,l}, \quad k, l = 0, 1, 2, \dots \quad (66)$$

**Theorem 7.1.** (Central limit theorem for Young diagrams) *According to (63), let*

$$u_k^{(n)} = \int_{\mathbb{R}} u_k(x) \Delta^{(n)}(dx), \quad k = 1, 2, \dots,$$

and let, as before,  $\xi_2, \xi_3, \dots$  stand for a system of independent standard Gaussians.

We have

$$\left\{ u_k^{(n)} \right\}_{k \geq 1} \xrightarrow{d} \left\{ \frac{\xi_{k+1}}{\sqrt{k+1}} \right\}_{k \geq 1}, \quad n \rightarrow \infty.$$

Recall that “ $\xrightarrow{d}$ ” means convergence in distribution.

Note that  $u_0^{(n)} \equiv 0$ , which explains why we start with  $k = 1$ , not  $k = 0$ . The theorem is proved at the end of the section. The scheme of the proof is as follows. We remark that the moments of  $\Delta^{(n)}$  (i.e., the random variables  $v^{(n)}$ , where the  $v$ 's are monomials) are expressed in terms of the elements  $\tilde{p}_k$ , appropriately centered and scaled. To evaluate the asymptotics of the corresponding random variables we employ Theorem 6.1. The main work reduces to expressing the (centered and scaled) elements  $\tilde{p}_k$  through the elements  $\eta_k$  and vice versa, up to lower degree terms.

As in Section 6, it is convenient to deal with the extended algebra  $\mathbb{A}^{\text{ext}} \supset \mathbb{A}$ . We extend the definition of  $\deg_1(\cdot)$  to  $\mathbb{A}^{\text{ext}}$  as explained in Section 6.

We introduce the elements  $q_1, q_2, \dots \in \mathbb{A}^{\text{ext}}$ , which are centered and scaled versions of the elements  $\tilde{p}_2, \tilde{p}_3, \dots$ :

$$q_k = \begin{cases} \frac{\tilde{p}_{k+1} - \frac{(2m)!}{m!m!} (p_1^\#)^m}{(k+1)(p_1^\#)^{k/2}}, & \text{if } k \text{ is odd, } k = 2m - 1, \text{ where } m = 1, 2, \dots, \\ \frac{\tilde{p}_{k+1}}{(k+1)(p_1^\#)^{k/2}}, & \text{if } k \text{ is even, } k = 2m, \text{ where } m = 1, 2, \dots \end{cases} \quad (67)$$

Since  $\frac{1}{2}\tilde{p}_2 = p_1$  and  $p_1 = p_1^\#$ , we have  $q_1 = 0$ .

**Proposition 7.2.** *For any  $\lambda \in \mathbb{Y}$ ,*

$$\int_{\mathbb{R}} x^k \Delta_\lambda(x) dx = \frac{q_{k+1}(\lambda)}{k+1}, \quad k = 0, 1, \dots \quad (68)$$

*Proof.* Set  $n = |\lambda|$ . By the definition of  $\Delta_\lambda(x)$ , see (62),

$$\int_{\mathbb{R}} x^k \Delta_\lambda(x) dx = \sqrt{n} \int_{\mathbb{R}} x^k \frac{\bar{\lambda}(x) - |x|}{2} dx - \sqrt{n} \int_{\mathbb{R}} x^k \frac{\Omega(x) - |x|}{2} dx.$$

By Proposition 2.2, for any  $k = 0, 1, \dots$ ,

$$\begin{aligned} \sqrt{n} \int_{\mathbb{R}} x^k \frac{\bar{\lambda}(x) - |x|}{2} dx &= n^{-(k+1)/2} \int_{\mathbb{R}} x^k \frac{\lambda(x) - |x|}{2} dx \\ &= \frac{1}{(k+1)(k+2)} \frac{\tilde{p}_{k+2}(\lambda)}{n^{(k+1)/2}} = \frac{1}{(k+1)(k+2)} \frac{\tilde{p}_{k+2}}{(p_1^\#)^{(k+1)/2}}(\lambda). \end{aligned}$$

By Propositions 2.2 and 5.3, for any  $k = 0, 1, \dots$ ,

$$\begin{aligned} \sqrt{n} \int_{\mathbb{R}} x^k \frac{\Omega(x) - |x|}{2} dx &= \frac{\sqrt{n} \tilde{p}_{k+2}[\Omega]}{(k+1)(k+2)} \\ &= \frac{\sqrt{n}}{(k+1)(k+2)} \cdot \begin{cases} \frac{(2m)!}{m!m!}, & \text{if } k \text{ is even, } k = 2m - 2, \text{ where } m = 1, 2, \dots, \\ 0, & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

Combining this with the definition of  $q_1, q_2, \dots$ , we get (68).

In order to apply Theorem 6.1 we need the expression of  $p_k^\#$  in terms of  $q_2, q_3, \dots$  within lower degree terms. We obtain this in two steps. First, using a trick, we deduce from Proposition 3.7 a formula expressing any  $q_k$  through  $p_2^\#, p_3^\#, \dots$ , up to lower degree terms. See Proposition 7.3. Next, we invert this formula, see Proposition 7.4. One could derive the result directly from Proposition 3.3 but this way turns out to be more difficult.

**Proposition 7.3.** *For any  $k = 2, 3, \dots$ ,*

$$q_k = \sum_{j=0}^{\lfloor \frac{k-2}{2} \rfloor} \binom{k}{j} \frac{p_{k-2j}^\#}{(p_1^\#)^{(k-2j)/2}} + \dots, \quad (69)$$

where the dots stand for a remainder term with  $\deg_1(\cdot) < 0$ .

Note that the elements occurring in the numerator of the right-hand side are  $p_2^\#, p_3^\#, \dots$  but not  $p_1^\#$ .

*Proof.* The claim of the proposition is equivalent to the following: for any  $k = 3, 4, \dots$ ,

$$\begin{aligned} \tilde{p}_k = & \sum_{j=0}^{\lfloor \frac{k-3}{2} \rfloor} \frac{k \downarrow j+1}{j!} p_{k-1-2j}^\# (p_1^\#)^j \\ & + \begin{cases} \frac{(2m)!}{m!m!} (p_1^\#)^m, & \text{if } k \text{ is even, } k = 2m, \\ 0, & \text{if } k \text{ is odd.} \end{cases} \\ & + \text{ terms with } \deg_1(\cdot) < k - 1. \end{aligned} \quad (70)$$

We shall deduce this from Proposition 3.7, which expresses  $\tilde{p}_k$  as a polynomial in  $p_1^\#, p_2^\#, \dots$ , up to terms of lower weight. A nontrivial point is how to switch from the weight filtration to the Kerov filtration.

Write the exact expansion of  $\tilde{p}_k$  through  $p_1^\#, p_2^\#, \dots$ ,

$$\tilde{p}_k = \sum_{\nu} a_{\nu} (p_1^\#)^{\nu_1} (p_2^\#)^{\nu_2} \dots,$$

where  $a_{\nu}$  are certain coefficients. Let us set

$$\|\nu\| = 2\nu_1 + 3\nu_2 + 4\nu_3 + \dots, \quad \|\nu\|' = 2\nu_1 + 2\nu_2 + 3\nu_3 + \dots,$$

so that

$$\|\nu\|' = \|\nu\| - (\nu_2 + \nu_3 + \dots). \quad (71)$$

We have

$$\text{wt}((p_1^\#)^{\nu_1} (p_2^\#)^{\nu_2} \dots) = \|\nu\|, \quad \deg_1((p_1^\#)^{\nu_1} (p_2^\#)^{\nu_2} \dots) \leq \|\nu\|'.$$

By Proposition 3.7, we know all coefficients  $a_{\nu}$  with the maximal value of  $\|\nu\|$  (it equals  $k$ ), while we need all coefficients with  $\|\nu\|' \geq k - 1$ . By (71),  $\|\nu\|'$  does not exceed  $k$ , and there are 3 possible cases:

- $\|\nu\|' = \|\nu\| = k$ . Then  $\nu_2 = \nu_3 = \dots = 0$ ,  $k = 2\nu_1$ , so that  $k$  is even; write it as  $2m$ . The corresponding monomial is  $(p_1^\#)^{k/2}$ .

- $\|\nu\|' = k - 1$ ,  $\|\nu\| = k$ . Then  $\nu_2 + \nu_3 + \dots = 1$ , i.e., exactly one of the numbers  $\nu_2, \nu_3, \dots$  equals 1. The corresponding monomial is of the form

$$p_{k-1-2j}^\# (p_1^\#)^j, \quad j = \nu_1 = 0, 1, \dots, \lfloor \frac{k-3}{2} \rfloor. \quad (72)$$

- $\|\nu\|' = \|\nu\| = k - 1$ . Then  $\nu_2 = \nu_3 = \dots = 0$ ,  $k$  is odd, and the corresponding monomial is  $(p_1^\#)^{(k-1)/2}$ .

In the first and second cases,  $\|\nu\|$  takes the maximal value  $k$ , and then the coefficients  $a_\nu$  are known from Proposition 3.7. In the third case,  $\|\nu\|$  is no longer maximal, so that Proposition 3.7 does not tell us what is the coefficient. However, an additional argument will imply that it is actually 0.

Indeed, in the third case  $k$  must be odd, which implies that  $\tilde{p}_k$  is antisymmetric with respect to “inv”, see (28). On the other hand, by virtue of (48), any monomial in  $p_1^\#, p_2^\#, \dots$  is either symmetric or antisymmetric. Therefore, in the expansion of  $\tilde{p}_k$  only antisymmetric monomials can occur. Since  $p_1^\#$  is symmetric, the monomial  $(p_1^\#)^{(k-1)/2}$  is also symmetric, so that it does not appear.

Thus, we have proved that the top degree component of  $\tilde{p}_k$  is obtained from the top weight terms as given in Proposition 3.7; we simply keep all terms proportional either to  $(p_1^\#)^{k/2}$  or to a monomial of the form (72), and remove all the remaining terms. This procedure leads to (70).

In the next proposition we invert (69).

**Proposition 7.4.** *For any  $k \geq 2$ ,*

$$\frac{p_k^\#}{(p_1^\#)^{k/2}} = \sum_{j=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^j \frac{k}{k-j} \binom{k-j}{j} q_{k-2j} + \dots, \quad (73)$$

where dots mean a remainder term with  $\deg_1(\cdot) < 0$ .

*Proof.* We employ the following combinatorial inversion formula, see [32], §2.4, (10):

Let  $a_0, a_1, a_2, \dots, b_0, b_1, b_2, \dots$  be formal variables. Then

$$\left\{ a_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{j} b_{k-2j} \right\}_{k=0,1,\dots} \iff \left\{ b_k = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \frac{k}{k-j} \binom{k-j}{j} a_{k-2j} \right\}_{k=0,1,\dots} \quad (74)$$

Set

$$a_0 = a_1 = 0, \quad a_k = q_k \quad (k \geq 2); \quad b_0 = b_1 = 0, \quad b_k = \frac{p_k^\#}{(p_1^\#)^{k/2}} \quad (k \geq 2).$$

The relations (69) coincide with the first system in (74), up to remainder terms of negative degree. These terms can be neglected, because they affect only similar remainder terms in the inverse relations. These inverse relations are given then by the second system in (74). This leads to (73).

*Proof of Theorem 7.1.* We rewrite (73) as follows. For any  $k \geq 2$ ,

$$\sum_{j=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^j \binom{k-1-j}{j} \frac{q_{k-2j}}{k-2j} = \frac{1}{k} \frac{p_k^\#}{(p_1^\#)^{k/2}} + R_k,$$

where  $R_k \in \mathbb{A}^{\text{ext}}$  is a certain element such that  $\deg_1(R_k) < 0$ .

In the left-hand side, we may extend the summation up to  $\lfloor \frac{k-1}{2} \rfloor$ , because  $q_1 = 0$ . Comparing this with formula (64) for  $u_{k-1}(x)$  and formula (68) for the moments of  $\Delta_\lambda$ , we conclude that

$$u_{k-1}^{(n)} = \frac{1}{\sqrt{k}} \eta_k^{(n)} + R_k^{(n)} \quad k \geq 2.$$

Or, equivalently,

$$u_k^{(n)} = \frac{1}{\sqrt{k+1}} \eta_{k+1}^{(n)} + R_{k+1}^{(n)} \quad k \geq 1.$$

As  $n \rightarrow \infty$ , the asymptotics of the (mixed) moments of the random variables  $u_1^{(n)}, u_2^{(n)}, \dots$  is the same as that for the random variables  $\frac{1}{\sqrt{k+1}} \eta_{k+1}^{(n)}$ ,  $k = 1, 2, \dots$ . Indeed, the remainder terms of negative degree do not affect the asymptotics, see Proposition 6.2. As for the moments of the random variables  $\eta_{k+1}^{(n)}$ , their asymptotics has been evaluated in the proof of Theorem 6.1. This concludes the proof.

## 8. The central limit theorem for transition measures of Young diagrams

Let  $\mathcal{M}$  denote the set of probability measures on  $\mathbb{R}$  with compact support, and let  $\mathcal{M}^0 \subset \mathcal{M}$  be the subset of measures with the first moment equal to 0.

**Proposition 8.1.** *There exists a bijective correspondence  $\omega \leftrightarrow \mu$  between  $\mathcal{D}$  and  $\mathcal{M}$ , which is also a bijection  $\mathcal{D}^0 \leftrightarrow \mathcal{M}^0$ . It is characterized by the relation*

$$\exp \int_{\mathbb{R}} \frac{\sigma'(x) dx}{x-z} = \int_{\mathbb{R}} \frac{\mu(dx)}{1 - \frac{x}{z}}, \quad (75)$$

where  $z \in \mathbb{C} \setminus I$ , where  $I \subset \mathbb{R}$  stands for a sufficiently large interval, and, as usual,  $\sigma(x) = \frac{1}{2}(\omega(x) - |x|)$ .

*Proof.* See [22], [24].

We call  $\mu$  the *transition measure* of the continual diagram  $\omega$ . In [24], the correspondence  $\sigma' \mapsto \mu$  defined by (75) is defined in a greater generality,

so that its range is the set of all (not necessarily compactly supported) probability measures on  $\mathbb{R}$ . (Note that in [24], the symbol  $\mathcal{M}$  refers to the latter set.)

Formula (75) means that the two sequences,

$$\left\{ -k \int_{\mathbb{R}} x^{k-1} \sigma'(x) dx \right\}_{k=1,2,\dots} \quad \text{and} \quad \left\{ \int_{\mathbb{R}} x^k \mu(dx) \right\}_{k=1,2,\dots},$$

are related to each other in exactly the same way as the two systems of generators of the algebra  $\Lambda$ ,  $\{\mathbf{p}_k\}_{k=1,2,\dots}$  and  $\{\mathbf{h}_k\}_{k=1,2,\dots}$ .

From now on and up to the end of this section we restrict ourselves to measures  $\mu$  from the subset  $\mathcal{M}^0$ .

**Definition 8.2.** Recall that the algebra  $\mathbb{A}$  can be realized as the image of  $\Lambda$  under the morphism (29), and let  $\tilde{h}_2, \tilde{h}_3, \dots$  denote the image in  $\mathbb{A}$  of the elements  $\mathbf{h}_2, \mathbf{h}_3, \dots$ .

We realize  $\mathbb{A}$  as an algebra of functions on  $\mathcal{M}^0$  by setting

$$\tilde{h}_k[\mu] = \int_{\mathbb{R}} x^k \mu(dx), \quad \mu \in \mathcal{M}^0, \quad k = 2, 3, \dots$$

Equivalently, for any  $f \in \mathbb{A}$ , we set  $f[\mu] = f[\omega]$ , where  $\omega \leftrightarrow \mu$  and  $f[\omega]$  is understood according to Definition 2.10 b).

**Proposition 8.3.** Let  $\omega = \lambda(\cdot)$ , where  $\lambda$  is a Young diagram, and let  $\{x_i\}_{i=1}^{m+1}$ ,  $\{y_j\}_{j=1}^m$  be the local extrema of  $\lambda(x)$ , see Section 2. Then the transition measure of  $\omega$  is supported by the finite set  $\{x_1, \dots, x_{m+1}\}$  and is given by the following formula:

$$\mu = \sum_{i=1}^{m+1} \mu_i \delta_{x_i}, \quad (76)$$

where  $\delta_x$  denotes the Dirac mass at  $x$  and the weights  $\mu_1, \dots, \mu_{m+1}$  are the coefficients in the expansion

$$\frac{\prod_{j=1}^m (z - y_j)}{\prod_{i=1}^{m+1} (z - x_i)} = \sum_{i=1}^{m+1} \frac{\mu_i}{z - x_i}. \quad (77)$$

*Proof.* See [22], [24].

The measure  $\mu$  defined by (76)–(77) is called the *transition measure* of a given Young diagram  $\lambda$ . For a justification of this term and more details, see [22], [23], [24].

**Proposition 8.4.** *The transition measure of  $\Omega \in \mathcal{D}^0$  is the “semi-circle distribution”  $\mu_{\text{s-c}}$  supported by  $[-2, 2]$ ,*

$$\mu_{\text{s-c}}(dx) = \frac{1}{2\pi} \sqrt{4 - x^2}. \quad (78)$$

*Proof.* See [22], [24].

**Definition 8.5.** *Fix  $n = 1, 2, \dots$ . To any  $\lambda \in \mathbb{Y}_n$  we assign a probability measure  $\widehat{\lambda} \in \mathcal{M}^0$  as follows:  $\widehat{\lambda}$  is the transition measure of the scaled diagram  $\bar{\lambda}(\cdot) \in \mathcal{D}^0$ . Equivalently,  $\widehat{\lambda}$  is the push-forward of the transition measure (76)–(77) under the shrinking  $x \mapsto n^{-1/2}x$  of the real axis.*

*Viewing  $\lambda$  as the random element of the probability space  $(\mathbb{Y}_n, M_n)$ , we interpret  $\widehat{\lambda}$  as a random probability measure.*

The next result is simply a reformulation of Theorem 5.5. Recall that, by Definition 8.2, we may view  $\mathbb{A}$  as a function algebra on  $\mathcal{M}^0$ .

**Theorem 8.6.** *(Law of large numbers for transition measures) As  $n \rightarrow \infty$ , the random measures  $\widehat{\lambda}$  concentrate near the Dirac mass at the element  $\mu_{\text{s-c}} \in \mathcal{M}^0$ , the semi-circle distribution (78).*

*In more detail, let  $\widehat{M}_n$  stand for the push-forward of the measure  $M_n$  under the correspondence  $\lambda \mapsto \widehat{\lambda}$ . Then*

$$\lim_{n \rightarrow \infty} \langle f, \widehat{M}_n \rangle = f[\mu_{\text{s-c}}], \quad \forall f \in \mathbb{A}.$$

*Proof.* Immediately follows from Theorem 5.5, Proposition 8.4 and Definition 8.2.

Now, our aim is to describe the fluctuations of the random measures  $\widehat{\lambda}$  around the semi-circle distribution  $\mu_{\text{s-c}}$ . We do not know if this can be achieved by a simple application of Theorem 7.1. The reason is that the transform  $\bar{\lambda}(\cdot) \mapsto \widehat{\lambda}$  is highly nonlinear. It turns out, however, that the proof of Theorem 7.1 can be readily translated to the language of transition measures: it suffices to deal with  $h_2, \tilde{h}_3, \dots$  instead of  $\tilde{p}_2, \tilde{p}_3, \dots$ .

The role of the polynomials  $u_k(x)$  is played now by the polynomials  $t_k(x)$ . These are slightly modified Chebyshev polynomials of the first kind. By definition,

$$\begin{aligned} t_k(x) = 2T_k\left(\frac{x}{2}\right) &= \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \frac{k}{k-j} \binom{k-j}{j} x^{k-2j} \\ &= k \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \frac{(k-1-j)!}{j!(k-2j)!} x^{k-2j}, \end{aligned}$$

where  $k = 1, 2, \dots$  and the  $T_k$ 's are the conventional Chebyshev polynomials, see [37], [9].

We also have (cf. (65), (66))

$$t_k(2 \cos \theta) = 2 \cos(k\theta), \quad k = 1, 2, \dots, \quad (79)$$

and

$$\int_{-2}^2 t_k(x)t_l(x) \frac{1}{2\pi \sqrt{4-x^2}} dx = \delta_{k,l}, \quad k, l = 1, 2, \dots \quad (80)$$

Given  $\lambda \in \mathbb{Y}_n$ , we set (cf. (62))

$$\widehat{\Delta}_\lambda = \sqrt{n}(\widehat{\lambda} - \mu_{s-c}). \quad (81)$$

This is a compactly supported measure on  $\mathbb{R}$  (in general, not a positive one). Dropping  $\lambda$ , which is viewed as the random element of  $(\mathbb{Y}_n, M_n)$ , we interpret (81) as a random measure, which we denote by  $\widehat{\Delta}^{(n)}$ .

Next, we set

$$t_k^{(n)} = \int_{\mathbb{R}} t_k(x) \widehat{\Delta}^{(n)}(dx).$$

This is a random variable, defined on the probability space  $(\mathbb{Y}_n, M_n)$ .

**Proposition 8.7.** *We have  $t_1^{(n)} = t_2^{(n)} \equiv 0$ .*

*Proof.* Recall that for any measure from  $\mathcal{M}^0$ , the first moment equals zero. In particular, this holds for  $\widehat{\lambda}$  and  $\mu_{s-c}$ , which implies  $t_1^{(n)} \equiv 0$ .

Next, the relation  $\mathbf{h}_2 = \frac{1}{2}(\mathbf{p}_1^2 + \mathbf{p}_2)$  in the algebra  $\Lambda$  turns, under the morphism (29), into the relation  $\widetilde{h}_2 = \widetilde{p}_2/2 = p_1$  in the algebra  $\mathbb{A}$ . It follows that  $\widetilde{h}_2[\lambda(\cdot)] \equiv n = |\lambda|$ , which implies  $\widetilde{h}_2[\widehat{\lambda}(\cdot)] \equiv 1$ . This in turn means that the second moment of  $\widehat{\lambda}$  equals 1. On the other hand, the second moment of  $\mu_{s-c}$  also equals 1. Therefore, the second moment of  $\widehat{\lambda} - \mu_{s-c}$  equals 0 for any  $\lambda \in \mathbb{Y}_n$ , so that  $\widetilde{h}_2^{(n)} \equiv 0$ .

**Theorem 8.8.** (Central limit theorem for transition measures, cf. Theorem 7.1)<sup>3</sup>

*In the notation introduced above,*

$$\left\{ t_k^{(n)} \right\}_{k \geq 3} \xrightarrow{d} \left\{ \sqrt{k-1} \xi_{k-1} \right\}_{k \geq 3},$$

where  $\xi_2, \xi_3, \dots$  are independent standard Gaussian random variables.

<sup>3</sup> As was already mentioned in Introduction, this result is due to the authors.

Here we start with  $k = 3$ , because  $t_1^{(n)} = t_2^{(n)} \equiv 0$ , see Proposition 8.5.

*Outline of proof.* Since the argument is strictly parallel to that given above for Theorem 7.1, we will not repeat all the details.

*Step 1: Expressing  $\tilde{h}_k$  through  $p_1^\#, p_2^\#, \dots$ , up to lower weight terms.*

This is a counterpart of Proposition 3.7. We start with formula (36) of Proposition 3.7, which we rewrite as follows

$$p_{k-1}^\# = -\frac{1}{k-1} [t^k] \{A^{-(k-1)}(t)\} + \dots, \quad k = 2, 3, \dots,$$

where

$$A(t) = 1 + \sum_{j=2}^{\infty} \tilde{h}_j t^j$$

and dots mean lower weight terms. Applying Proposition 3.6 we invert this formula and get

$$\tilde{h}_k = \frac{1}{k+1} [u^k] \{B^{k+1}(u)\} + \dots, \quad k = 2, 3, \dots,$$

where

$$B(u) = 1 + \sum_{j=2}^{\infty} p_{j-1}^\# u^j$$

and dots mean a polynomial in  $\{p_{j-1}^\#\}$  of weight  $< k$ , where, by definition,  $\text{wt}(p_{j-1}^\#) = j$ . More explicitly,

$$\tilde{h}_k = \sum_{\substack{m_2, m_3, \dots \\ 2m_2 + 3m_3 + \dots = k}} \frac{k \downarrow (\sum m_i - 1)}{\prod m_i!} \prod (p_{i-1}^\#)^{m_i} + \text{lower weight terms.} \quad (82)$$

*Step 2: Switching to the Kerov filtration.*

The same argument as that used in the proof of Proposition 7.3 makes it possible to derive from (82) the following expression, cf. (70).

$$\begin{aligned} \tilde{h}_k &= \sum_{j=0}^{\lfloor \frac{k-3}{2} \rfloor} \binom{k}{j} p_{k-1-2j}^\# (p_1^\#)^j \\ &+ \begin{cases} \frac{1}{m+1} \binom{2m}{m} (p_1^\#)^m, & \text{if } k \text{ is even, } k = 2m, m = 1, 2, \dots, \\ 0, & \text{if } k \text{ is odd} \end{cases} \\ &+ \text{terms with } \deg_1(\cdot) < k - 1. \end{aligned}$$

Define the elements  $g_k \in \mathbb{A}^{\text{ext}}$  by (cf. (67))

$$g_k = \begin{cases} \frac{\tilde{h}_k - \frac{1}{m+1} \binom{2m}{m} (p_1^\#)^m}{(p_1^\#)^{(k-1)/2}}, & \text{if } k = 2m, m = 1, 2, \dots, \\ \frac{\tilde{h}_k}{(p_1^\#)^{(k-1)/2}}, & \text{if } k \text{ is odd.} \end{cases} \quad (83)$$

The above expression for  $\tilde{h}_k$  is equivalent to

$$g_k = \sum_{j=0}^{\lfloor \frac{k-3}{2} \rfloor} \binom{k}{j} \frac{p_{k-1-2j}^\#}{(p_1^\#)^{(k-1-2j)/2}} + \dots, \quad k = 3, 4, \dots, \quad (84)$$

where dots mean terms with  $\deg_1(\cdot) < 0$ .

*Step 3: Inverse formula expressing  $p_{k-1}^\# / (p_1^\#)^{(k-1)/2}$  through  $g_3, g_4, \dots$*

This is a counterpart of Proposition 7.4. We note that (84) is quite similar to (69). Exactly as in Proposition 7.4, we get

$$\frac{p_{k-1}^\#}{(p_1^\#)^{(k-1)/2}} = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \frac{k}{k-j} \binom{k-j}{j} g_{k-2j} + \dots, \quad (85)$$

where, by convention,  $g_0 = g_1 = g_2 = 0$  and dots mean terms with  $\deg_1(\cdot) < 0$ .

*Step 4: Interpretation in terms of Chebyshev's polynomials.*

This final step is similar to the proof of Theorem 7.1 at the end of Section 7. The moments of the semi-circle distribution have the following form, cf. Proposition 5.3:

$$\int_{-2}^2 x^k \mu_{\text{s-c}}(dx) = \begin{cases} \frac{1}{m+1} \binom{2m}{m}, & \text{if } k = 2m, m = 1, 2, \dots, \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

From this, the definition of the elements  $g_k$  (see (83)), and Definition 8.5 we get

$$g_k(\lambda) = \sqrt{n} \int_{-\infty}^{\infty} x^k (\hat{\lambda} - \mu_{\text{s-c}})(dx) = \int_{-\infty}^{\infty} x^k \hat{\Delta}_\lambda(dx), \quad k = 3, 4, \dots \quad (86)$$

It follows from (86) that any polynomial  $v \in \mathbb{R}[x]$  may be identified with an element of  $\mathbb{A}^{\text{ext}}$  (say,  $\hat{v}$ ) via

$$\hat{v}(\lambda) = \int_{\mathbb{R}} v(x) \hat{\Delta}_\lambda(dx), \quad \lambda \in \mathbb{Y}.$$

Or, equivalently,

$$\mathbb{R}[x] \ni v = \sum_i c_i x^i \quad \longleftrightarrow \quad \hat{v} = \sum_i c_i g_i \in \mathbb{A}^{\text{ext}}.$$

In particular, for  $v = t_k$  we get from (79)

$$\hat{t}_k = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \frac{k}{k-j} \binom{k-j}{j} g_{k-2j}, \quad k = 3, 4, \dots$$

Recall also that  $\hat{t}_0 = \hat{t}_1 = \hat{t}_2 = 0$ .

Comparing this with (85) we see that

$$\hat{t}_k = \frac{p_{k-1}^{\#}}{(p_1^{\#})^{(k-1)/2}} + R_{k-1}, \quad R_{k-1} \in \mathbb{A}^{\text{ext}}, \quad \deg_1(R_{k-1}) < 0, \quad k = 3, 4, \dots$$

Or by the definition of the elements  $\eta_k$ , see (57),

$$\hat{t}_k = \sqrt{k-1} \eta_{k-1} + R_{k-1}, \quad k = 3, 4, \dots,$$

so that

$$t_k^{(n)} = \sqrt{k-1} \eta_{k-1}^{(n)} + R_{k-1}^{(n)}, \quad k = 3, 4, \dots$$

Then the proof is completed as at the end of Section 7.

## 9. Discussion

Recall that a *generalized Gaussian process* is a Gaussian measure in a space  $\mathcal{F}'$  of distributions (=generalized functions);  $\mathcal{F}'$  is supposed to be the dual to a space  $\mathcal{F}$  of test functions. Given a test function  $\varphi \in \mathcal{F}$ , the result of its pairing with the random distribution defined by the process is a random Gaussian variable. See, e.g., Gelfand–Vilenkin [12], Simon [35].

We shall define Gaussian processes via expansions in some orthogonal systems of functions with random coefficients (a useful general reference on such random series is Kahane's book [20]). Consider the random series

$$\Delta(x) = \sum_{k=1}^{\infty} \frac{\xi_{k+1} u_k(x) \sqrt{4-x^2}}{2\pi \sqrt{k+1}}, \quad -2 \leq x \leq 2. \quad (87)$$

Here, as above,  $\xi_2, \xi_3, \dots$  are independent standard Gaussian random variables and  $u_1, u_2, \dots$  are modified Chebyshev's polynomials of the second kind (see (64), (65)). The series (87) correctly defines a generalized Gaussian process, where as  $\mathcal{F}'$  we take the space  $(C^\infty(\mathbb{R}))'$  of compactly supported

distributions on the real line. But the process actually lives on the subspace of distributions concentrated on  $[-2, 2]$ ). For any test function  $\varphi \in \mathcal{F} = C^\infty(\mathbb{R})$ ,

$$\langle \varphi, \Delta \rangle = \int_{-2}^2 \varphi(x) \Delta(x) dx = \sum_{k=1}^{\infty} \frac{\xi_{k+1}}{2\pi\sqrt{k+1}} \int_{-2}^2 \varphi(x) u_k(x) \sqrt{4-x^2} dx$$

is a Gaussian random variable. In particular, setting  $\varphi(x) = u_k(x)$  we get, by the orthogonality relation (66),

$$\langle u_k, \Delta \rangle = \frac{\xi_{k+1}}{\sqrt{k+1}}, \quad k = 1, 2, \dots$$

Informally, the result of Theorem 7.1 can be stated as follows: for the random Plancherel diagram  $\lambda \in \mathbb{Y}_n$ ,

$$\bar{\lambda}(x) \sim \Omega(x) + \frac{2}{\sqrt{n}} \Delta(x), \quad n \rightarrow \infty, \quad (88)$$

where  $\bar{\lambda}(x)$  was introduced in Definition 2.3, and  $\Delta(x)$  is given by (87).

Next, consider the random series

$$\hat{\Delta}(x) = \sum_{k=3}^{\infty} \frac{\sqrt{k-1} \xi_{k-1} t_k(x)}{2\pi\sqrt{4-x^2}}, \quad -2 < x < 2. \quad (89)$$

Here  $\xi_2, \xi_3, \dots$  are as above and  $t_3, t_4, \dots$  are modified Chebyshev's polynomials of the first kind (see (79), (79)). The series (89) correctly defines a generalized Gaussian process on the same space  $\mathcal{F}' = (C^\infty(\mathbb{R}))'$  of compactly supported distributions. For any test function  $\varphi \in C^\infty(\mathbb{R})$ ,

$$\langle \varphi, \hat{\Delta} \rangle = \int_{-2}^2 \varphi(x) \hat{\Delta}(x) dx = \sum_{k=3}^{\infty} \frac{\sqrt{k-1} \xi_{k-1}}{2\pi} \int_{-2}^2 \frac{\varphi(x) u_k(x)}{\sqrt{4-x^2}} dx$$

is a Gaussian random variable. In particular, setting  $\varphi(x) = t_k(x)$  we get, by the orthogonality relation (80),

$$\langle t_k, \hat{\Delta} \rangle = \sqrt{k-1} \xi_{k-1}, \quad k = 3, 4, \dots$$

Informally, the result of Theorem 8.6 can be stated as follows: for the random Plancherel diagram  $\lambda \in \mathbb{Y}_n$ ,

$$\hat{\lambda}(x) \sim \mu_{s-c}(x) + \frac{1}{\sqrt{n}} \hat{\Delta}(x), \quad n \rightarrow \infty, \quad (90)$$

where the transition measure  $\hat{\lambda}$  (see Definition 8.5) is viewed as a generalized function, and  $\hat{\Delta}(x)$  is given by (89).

Let us compare these results with the central limit theorem for the Gaussian unitary ensemble. Consider the space  $\mathbb{H}_N$  of  $N \times N$  complex Hermitian matrices, and equip it with the Gaussian measure

$$\text{Gauss}_N(dX) = \left(\frac{N}{2\pi}\right)^{N^2/2} \exp\left\{-\frac{N}{2} \text{tr}(X^2)\right\} \text{Leb}(dX), \quad (91)$$

where  $X$  ranges over  $\mathbb{H}_N$  and  $\text{Leb}$  denotes the Lebesgue measure on  $\mathbb{H}_N \simeq \mathbb{R}^{N^2}$ .

To any matrix  $X \in \mathbb{H}_N$  we assign a certain probability measure  $\mu_X$  on  $\mathbb{R}$ , which we prefer to view as a generalized function:

$$\mu_X(x) = \frac{1}{N}(\delta(x - x_1) + \dots + \delta(x - x_N)), \quad (92)$$

where  $x_1, \dots, x_N$  are the eigenvalues of  $X$ , and  $\delta(x)$  is the delta function. Dropping  $X$ , which we view as the random element of the probability space  $(\mathbb{H}_N, \text{Gauss}_N)$ , we regard (92) as the random generalized function  $\mu(x)$ . Then we have the following *central limit theorem for the Gaussian unitary ensemble* (we state it informally):

$$\mu(x) \sim \mu_{\text{s-c}}(x) + \frac{1}{N} \tilde{\Delta}(x), \quad N \rightarrow \infty, \quad (93)$$

where  $\tilde{\Delta}(x)$  is the Gaussian process on  $[-2, 2]$  defined by the random series

$$\tilde{\Delta}(x) = \sum_{k=1}^{\infty} \frac{\sqrt{k} \xi_k t_k(x)}{2\pi\sqrt{4-x^2}} \quad (94)$$

with independent standard Gaussians  $\xi_1, \xi_2, \dots$ .

For the rigorous formulation and proof of this result (and its generalizations), see [17]. Note that similar results hold for other random matrix ensembles, see [7], [8], [16]. As explained in [16], [17], this subject has close links with the famous Szegő theorem on asymptotics of Toeplitz determinants.

Comparing (89) and (94) we see that the Gaussian processes  $\hat{\Delta}(x)$  and  $\tilde{\Delta}(x)$  look rather close. Another observation is that

$$-\frac{1}{2} \cdot \frac{d}{dx} \Delta(x) = \sum_{k=2}^{\infty} \frac{\sqrt{k} \xi_k t_k(x)}{2\pi\sqrt{4-x^2}}. \quad (95)$$

That is, the derivative of the process  $\Delta(x)$  coincides, up to factor  $-1/2$  and the first term, with the process  $\tilde{\Delta}(x)$ . This is readily seen from the following formulas.

All the three series  $\Delta(x)$ ,  $\widehat{\Delta}(x)$ ,  $\widetilde{\Delta}(x)$  look especially simply after change of a variable,  $x = 2 \cos \theta$ . Using (65), (79) we get

$$\begin{aligned}\Delta(2 \cos \theta) &= \frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\xi_k}{\sqrt{k}} \sin(k\theta), \\ \widehat{\Delta}(2 \cos \theta) &= \frac{1}{2\pi} \sum_{k=3}^{\infty} \frac{\sqrt{k-1} \xi_{k-1} \cos(k\theta)}{\sin \theta}, \\ \widetilde{\Delta}(2 \cos \theta) &= \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\sqrt{k} \xi_k \cos(k\theta)}{\sin \theta}.\end{aligned}$$

## 10. Free cumulants and Biane's theorem

Let, as above,

$$H(t) = 1 + \sum_{j=1}^{\infty} \mathbf{h}_j t^j \in \Lambda[[t]]$$

be the generating series for the complete homogeneous symmetric functions. We introduce elements  $\mathbf{f}_1, \mathbf{f}_2, \dots$  in  $\Lambda$  as follows:

$$\mathbf{f}_1 = \mathbf{h}_1; \quad \mathbf{f}_k = -\frac{1}{k-1} [t^k] \left\{ H^{-(k-1)}(t) \right\}, \quad k = 2, 3, \dots$$

More explicitly,

$$\mathbf{f}_k = \sum_{\substack{m_1, m_2, \dots \geq 0 \\ 1 \cdot m_1 + 2 \cdot m_2 + \dots = k}} (-1)^{\sum m_j - 1} k^{\uparrow(\sum m_j - 1)} \prod_{j \geq 1} \frac{\mathbf{h}_j^{m_j}}{m_j!},$$

where

$$x^{\uparrow m} = \frac{\Gamma(x+m)}{\Gamma(x)} = x(x+1) \dots (x+m-1).$$

This definition is inspired by Voiculescu's free probability theory [41], [42]. Let  $\mu$  be a compactly supported probability measure on  $\mathbb{R}$ , i.e., an element of  $\mathcal{M}$ , in our notation. When the  $\mathbf{h}_k$ 's are specialized to the moments of  $\mu$ ,

$$\mathbf{h}_k \longrightarrow \int_{\mathbb{R}} x^k \mu(dx), \quad k = 1, 2, \dots,$$

the elements  $\mathbf{f}_k$  turn into the *free cumulants* of the measure  $\mu$ . The free cumulants are counterparts of the semi-invariants in the sense of conventional

probability theory. The free cumulants are additive functionals with respect to additive free convolution of measures (just as the semi-invariants are additive functionals with respect to the conventional convolution product). See [41], [42], [36].

Denote by  $\tilde{f}_k \in \mathbb{A}$  the image of  $\mathbf{f}_k \in \Lambda$  under the morphism (29). Note that  $\tilde{f}_1 = 0$ . Let  $\lambda \in \mathbb{Y}$  be arbitrary and let  $\mu \in \mathcal{M}^0$  be the transition measure of  $\lambda$ , see (76)–(77). Then  $\tilde{f}_k(\lambda)$  coincides with the  $k$ th free cumulant of  $\mu$ .

**Proposition 10.1.** *For any  $k = 1, 2, \dots$ , the element  $\tilde{f}_{k+1}$  coincides with the top weight homogeneous component of  $p_k^\#$ .*

*Proof.* By the very definition

$$\tilde{f}_{k+1} = -\frac{1}{k} [t^{k+1}] \left\{ \left( 1 + \sum_{j \geq 2} \tilde{h}_j t^j \right)^{-k} \right\} = -\frac{1}{k} [t^{k+1}] \left\{ \exp\left(-k \sum_{j \geq 2} \frac{\tilde{p}_j}{j} t^j\right) \right\},$$

which is exactly formula (34). Then the claim follows from Proposition 3.5.

Biane [4] found out that free cumulants emerge in the asymptotic theory of characters of the symmetric groups. To state his result we need a notation.

Given  $A > 1$ , let  $\mathbb{Y}(A)$  denote the set of the Young diagrams  $\lambda$  such that  $\lambda_1 \leq \sqrt{n}A$ ,  $\lambda'_1 \leq \sqrt{n}A$ , where  $n = |\lambda|$ . Equivalently,  $\bar{\lambda}(x) = |x|$  whenever  $|x| \geq A$ . Recall that  $\bar{\lambda}(x) = n^{-1/2}(n^{1/2}x)$  is the scaled version of  $\lambda(\cdot)$ .

**Theorem 10.2.** ([4], Th. 1.3) *Fix an arbitrary  $A > 1$ . For any partition  $\rho$  and any  $\lambda \in \mathbb{Y}(A)$ ,*

$$\frac{\chi_{\rho \cup 1^{n-|\rho|}}^\lambda}{\dim \lambda} = n^{-\frac{|\rho| - \ell(\rho)}{2}} \prod_{j \geq 1} \tilde{f}_{j+1}^{m_j(\rho)}[\bar{\lambda}(\cdot)] + O\left(n^{-\frac{|\rho| - \ell(\rho)}{2} - 1}\right), \quad (96)$$

where  $n = |\lambda|$  is assumed to be  $\geq |\rho|$ . Here the estimate of the remainder term depends only on  $A$  and  $\rho$ , and is uniform on  $\lambda$  provided that  $\lambda$  ranges over  $\mathbb{Y}(A)$ .

*Comments.* 1) All terms in (96) do not depend on  $m_1(\rho)$ . Indeed, this is evident for  $\rho \cup 1^{n-|\rho|}$  and  $|\rho| - \ell(\rho)$ . On the other hand  $\tilde{f}_2(\bar{\lambda}(\cdot)) = 1$ , so that the factor corresponding to  $j = 1$  equals 1.

2) As pointed out by Biane, formula (96) implies that if  $\{\lambda\}$  is a sequence of diagrams in  $\mathbb{Y}(A)$  such that  $n = |\lambda| \rightarrow \infty$  and  $\bar{\lambda}(\cdot)$  uniformly converges

to a continual diagram  $\omega \in \mathcal{D}^0$  then

$$\frac{\chi_{\rho \cup 1^{n-|\rho|}}^\lambda}{\dim \lambda} \sim C n^{-\frac{|\rho|-\ell(\rho)}{2}}, \quad C = \prod_{j \geq 1} \tilde{f}_{j+1}^{m_j(\rho)}[\omega]. \quad (97)$$

Note that in some cases the constant  $C$  can vanish, which implies a faster decay of the character values: this happens, for instance, when  $\omega = \Omega$  and  $\rho$  is nontrivial (i.e. distinct from  $(1^r)$ ), because  $\tilde{f}_k[\Omega] = 0$  for any  $k \geq 3$ .

3) In this result, the assumption that  $\lambda$  ranges over a set of the form  $\mathbb{Y}(A)$  plays a key role. When this assumption is dropped, quite a different estimate for the left-hand side of (96) holds, see Roichman's paper [33].

4) Biane [5] obtained further results in this direction.

We shall give an alternative proof of this Biane's theorem. Our argument seems to be rather simple and transparent.

*Proof of Theorem 10.2.* Multiply both sides of (96) by

$$n^{\downarrow|\rho|} = n^{|\rho|} (1 + O(n^{-1})).$$

Then, by Definition 4.1, (96) is transformed to

$$p_\rho^\#(\lambda) = n^{\frac{|\rho|+\ell(\rho)}{2}} \prod_{j \geq 1} \tilde{f}_{j+1}^{m_j(\rho)}[\bar{\lambda}(\cdot)] + O\left(n^{\frac{|\rho|+\ell(\rho)}{2}-1}\right), \quad (98)$$

Expand  $p_\rho^\#$  into the sum of its weight homogeneous components:

$$p_\rho^\# = \sum_{j=0}^{|\rho|+\ell(\rho)} F_\rho^{(j)}, \quad \text{wt}(F_\rho^{(j)}) = |\rho| + \ell(\rho) - j, \quad (99)$$

so that  $F_\rho^{(0)}$  is the top weight component. By Proposition 4.9,

$$F_\rho^{(0)} = \prod_{j \geq 1} (\text{the top weight component of } p_j^\#)^{m_j(\rho)}.$$

Hence, by Proposition 10.1,

$$F_\rho^{(0)} = \prod_{j \geq 1} \tilde{f}_{j+1}^{m_j(\rho)}. \quad (100)$$

By virtue of (28), any weight homogeneous element of  $\mathbb{A}$  is either symmetric or antisymmetric with respect to “inv”, depending on whether its weight is even or odd. It follows from (47) that the element  $p_\rho^\#$  is either symmetric or antisymmetric, depending on the parity of the number  $|\rho| + \ell(\rho)$ . It follows that, in the expansion (99), we have  $F_\rho^{(j)} = 0$  for all odd  $j$ .

Using this and applying Proposition 2.11, we get from (99)

$$p_{\rho}^{\#}(\lambda) = \sum_{j=0}^{[(|\rho|+\ell(\rho))/2]} n^{\frac{|\rho|+\ell(\rho)}{2}-j} F_{\rho}^{(2j)}[\bar{\lambda}(\cdot)]. \quad (101)$$

Set  $\omega = \bar{\lambda}$  and let, as usual,  $\sigma = (\omega - |\cdot|)/2$ . By virtue of the assumption  $\lambda \in \mathbb{Y}(A)$ , the support of  $\sigma$  is contained in  $[-A, A]$ . It follows that  $|\tilde{p}_k[\omega]| \leq 2A^k$  for any  $k \geq 2$  (to see this, apply (13) and the general estimate  $|\sigma'(\cdot)| \leq 1$ ). Hence for any element  $F \in \mathbb{A}$  we get the estimate

$$|F[\bar{\lambda}(\cdot)]| \leq \text{Const}, \quad \lambda \in \mathbb{Y}(A), \quad (102)$$

where the constant depends only on  $A$  and the degree of  $F$  as a polynomial in  $\tilde{p}_2, \tilde{p}_3, \dots$ . Applying the estimate (102) to the terms of the expansion (101) and taking into account (100) we get the required formula (96).

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